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SOME ASPECTS OF INTUITIONISTIC  
AND CLASSICAL SEMANTICS

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ABSTRACT

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SOME ASPECTS OF INTUITIONISTIC AND CLASSICAL SEMANTICS.

The existence of truth-table systems for the classical and the intuitionistic propositional calculi is discussed.

Three methods of search for counterexamples to each non-valid formula in the classical propositional, the classical predicate and the intuitionistic propositional calculi are developed.

The decidability of both the classical and the intuitionistic propositional calculi is proved. In each case, two proofs are given: one is proof-theoretic; and the other is model-theoretic.

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## INTRODUCTION

The aim of this thesis is to discuss some of the basic aspects of the classical and intuitionistic semantics.

In the first chapter, we discuss the existence of a truth-table system for the classical and the intuitionistic propositional calculi. In the case of the classical propositional calculus, we prove the existence of a truth-table system with two truth values, then we use Godel's counterexample to show the non-existence of truth-table systems with finitely many truth values for the intuitionistic propositional calculus, but we shall be able to prove the existence of one with infinitely many truth values.

In the second chapter, we develop three methods which give us counterexamples to each non-valid formula in the classical propositional, the classical predicate and the intuitionistic propositional calculi.

In the third and the final chapter, we prove the decidability of both the classical and the intuitionistic propositional calculi. In each case, we give two proofs: one is proof theoretic, and the other is model theoretic.

The bibliography will include the references which are used in the thesis, together with some other references in which valuable discussions of some of the topics that are exposed in this thesis may be found.

Usually, a thesis' introduction ends with the student thanking his supervisor, but I shall not do so since there are no words that could express my gratefulness to Professor M.E. Szabo for his continued academic and personal support.

## CHAPTER I

### TRUTH TABLES FOR CLASSICAL AND INTUITIONISTIC PROPOSITIONAL CALCULI

#### 1.1 Introduction.

A formal logical system  $LS$  has a truth-table system if there exists a model  $A$  such that any formula  $X$ , which belongs to the language of  $LS$ , is valid in the semantics of  $LS$ , if and only if it is valid in  $A$ , and in this case we call the elements of  $A$  the truth values of the truth-table system.

In this chapter, we show the existence of a truth-table system for the classical propositional calculus with two truth values, and the existence of a truth-table system for the intuitionistic propositional calculus with infinitely many truth values. We also show the non-existence of any truth table system for the intuitionistic propositional calculus with finitely many truth values.

#### 1.2 Language.

Throughout this chapter, our alphabet will be a countable set of propositional variables  $p_1, p_2, \dots, q_1, q_2, \dots$ , three binary connectives  $\wedge, \vee, \supset$ , and one unary connective  $\sim$ , together with left and right parentheses  $(, )$ .



The notion of a formula is given recursively by the following rules:

F0. If  $P$  is a propositional variable,  $P$  is a formula.

F1. If  $X$  is a formula, so is  $\sim X$ .

F2,3,4. If  $X$  and  $Y$  are formulae, so are  $(X \wedge Y)$ ,  $(X \vee Y)$ ,  $(X \supset Y)$ .

A propositional variable will sometimes be called an atomic formula.

We shall omit writing outer parentheses in a formula when no confusion can result. We shall use  $X$ ,  $Y$ , and  $Z$  to represent any formula.

The notion of immediate subformula is given by the following rules:

IS0.  $P$  has no immediate subformula.

IS1.  $\sim X$  has exactly one immediate subformula, namely  $X$ .

IS2,3,4.  $(X \wedge Y)$ ,  $(X \vee Y)$ ,  $(X \supset Y)$  each have exactly two immediate subformulae, namely  $X$  and  $Y$ .

The notion of a subformula is defined as follows:

S0.  $X$  is a subformula of  $X$ .

S1. If  $X$  is an immediate subformula of  $Y$ , then  $X$  is a subformula of  $Y$ .

S2. If  $X$  is a subformula of  $Y$  and  $Y$  is a subformula of  $Z$ , then  $X$  is a subformula of  $Z$ .

### 1.3 Algebraic Models for Classical Propositional Calculus and Validity.

In this section, we state the algebraic semantics of the classical propositional calculus. [In the subsequent we shall denote the classical propositional calculus by CPC].

Definition 1.3.1: A Boolean algebra is a pair  $\langle B, \leq \rangle$  where  $B$  is a non-empty set and  $\leq$  is a partial ordering relation on  $B$  such that the following conditions are satisfied:

B0. If  $a, b \in B$ , then the least upper bound  $(a \vee b)$  and the greatest lower bound  $(a \wedge b)$  exist.

B1. If  $a, b, c \in B$ , then we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad (1)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad (2)$$

B2.  $B$  contains a maximal element  $1$  and a minimal element  $0$ .

B3. If  $a \in B$  then  $B$  contains an element  $a^*$ , such that

$$a \wedge a^* = 0 \quad \text{and} \quad a \vee a^* = 1.$$

Equations (1) and (2) are called the distributive laws and

$a^*$  is called the complement of  $a$ . It can be proved that

if  $a \in B$  then  $a^*$  is unique. We shall refer to a Boolean algebra by its underlying set when no confusion can result.

A thorough treatment of Boolean algebras may be found in

[B.&S.] or in [R.&S].

Definition 1.3.2:  $f$  is called a homomorphism [from the set  $F$  of formulae to the Boolean algebra  $B$ ] if

$f: F \rightarrow B$  and

$$(1) \quad f(X \wedge Y) = f(X) \wedge f(Y)$$

$$(2) \quad f(X \vee Y) = f(X) \vee f(Y)$$

$$(3) \quad f(\sim X) = (f(X))^*$$

$$(4) \quad f(X \supset Y) = f(X) \supset f(Y) \quad \text{where} \quad f(X) \supset f(Y) = (f(X))^* \vee f(Y)$$

Notice that on the right hand side of (1) and (2) the symbols " $\wedge$ ", " $\vee$ " are the inf. and sup. operators of the Boolean algebra, whereas on the left hand side they are symbols of our alphabet.

If  $B$  is a Boolean algebra and  $f$  is a homomorphism, the pair  $(B, f)$  is called an (algebraic) model for CPC. If  $X$  is a formula,  $X$  is called valid in the model  $(B, f)$  if  $f(X) = 1$  and we write  $B \models_f X$ .  $X$  is called valid if  $X$  is valid in every model, and we write  $\models X$ .

#### 1.4 The Lindenbaum Algebra of CPC.

First we define a relation  $\equiv$  on  $F$  by

$$X \equiv Y \text{ iff } \vdash X \supset Y \text{ and } \vdash Y \supset X$$

where  $\vdash X \supset Y$  means that  $(X \supset Y)$  is provable (See p.14.).

It can be proved that  $\equiv$  is an equivalence relation on  $F$ . Let

$|X|$  be the equivalence class generated by  $X$  under  $\equiv$  and let

$$L = \{|X| : X \in F\}$$

and define the relation  $\leq$  on  $L$  by

$$|X| \leq |Y| \text{ iff } \vdash X \supset Y$$

It is easy to prove that  $\langle L, \leq \rangle$  is a Boolean algebra and in this Boolean algebra

$$(1) \quad |X| \wedge |Y| = |X \wedge Y|$$

$$(2) \quad |X| \vee |Y| = |X \vee Y|$$

$$(3) \quad |X| \supset |Y| = |X \supset Y|$$

$$(4) \quad (|X|)^* = |\sim X|$$

$$(5) \quad |X| = 1 \text{ iff } \vdash X \text{ and } |X| = 0 \text{ iff } \vdash \sim X$$

We call  $\langle L, \leq \rangle$  the Lindenbaum algebra of CPC.

### 1.5. Truth-Table System for CPC.

In this section, we are to show that a formula  $X$  is valid if and only if it is valid in every model whose first component is  $2$  (the two points Boolean algebra). That proves the existence of a truth-table system for CPC with two truth values, namely  $0, 1$  (the members of  $2$ ). To do so, we need the definition of a Boolean algebra homomorphism.

Definition 1.5.1: If  $A$  and  $B$  are Boolean algebras, and  $f$  is a function from  $A$  to  $B$ , then  $f$  is said to be a

Boolean algebra homomorphism if the following conditions are satisfied:

- (1)  $f(x \wedge y) = f(x) \wedge f(y)$  for all  $x, y \in A$
- (2)  $f(x \vee y) = f(x) \vee f(y)$  for all  $x, y \in A$
- (3)  $f(x^*) = (f(x))^*$  for all  $x \in A$ .

Let  $f$  be a homomorphism from  $F$  to a Boolean algebra  $B$ . We have the following lemma.

Lemma 1.5.1. The function  $f': L \rightarrow B$  whose value at  $|X|$  is equal to  $f(X)$ , for every  $|X| \in L$ , is a Boolean algebra homomorphism.

Proof. First, we notice that  $f'$  is well defined since if  $|X| = |Y|$ , then we have

$$\vdash X \supset Y \text{ and } \vdash Y \supset X$$

But CPC is complete. Hence

$$\models X \supset Y \text{ and } \models Y \supset X$$

This implies that

$$f(X \supset Y) = f(Y \supset X) = 1$$

Using the fact that  $f$  is a homomorphism from  $F$  to  $B$  we get,

$$f(X) \supset f(Y) = f(Y) \supset f(X) = 1$$

Hence

$$f(X) * \vee f(Y) = 1 \quad (\text{By the definition of } \Rightarrow)$$

and so

$$\begin{aligned} f(X) &= f(X) \wedge 1 \\ &= f(X) \wedge ((f(X)) * \vee f(Y)) \\ &= f(X) \wedge f(Y) \leq f(Y) \end{aligned}$$

By the same argument, we can prove that

$$f(Y) \leq f(X)$$

Thus  $f(Y) = f(X)$ , and  $f'$  is well defined.

Now, we shall show that  $f'$  satisfies conditions

(1), (2) and (3) of definition 1.5.1.

$$\begin{aligned} (1) \quad f'(|X| \wedge |Y|) &= f'(|X \wedge Y|) \\ &= f(X \wedge Y) \\ &= f(X) \wedge f(Y) \\ &= f'(|X|) \wedge f'(|Y|) \end{aligned}$$

(2) Same as in one.

$$\begin{aligned} (3) \quad f'(|X| *) &= f'(|\sim X|) \\ &= f(\sim X) \\ &= (f(X)) * \\ &= (f|X|) * \end{aligned}$$

This completes the proof of the lemma.

Now, we shall introduce the definitions of filters and ultrafilters, which we are going to use in the proof of the following theorem.

Definition 1.5.2: A filter in a Boolean algebra  $B$  is a non-empty subset  $A$  of  $B$  which satisfies:

- (1) For all  $x, y \in A$ ,  $x \wedge y \in A$ .
- (2) For  $x \in A$  and  $y \in B$ , if  $x \leq y$  then  $y \in A$ .

Definition 1.5.3: An ultrafilter is a proper filter which has no proper extensions which are also proper filters.

Theorem 1.5.1. A formula  $X$  is valid if and only if for every homomorphism  $f$  from  $F$  to  $2$ ,  $2 \models_f X$ .

Proof. The "only if" part is an immediate consequence of the definitions of validity. To prove the "if" part, we assume that  $X$  is not valid. Hence there exists a Boolean algebra  $B$  and a homomorphism  $f$  such that  $f(X) \neq 1$ . So we can find an ultrafilter  $U$  in  $B$  such that  $f(X) \notin U$  but  $1 \in U$  (See [B.&S.]).

We define  $g: B \rightarrow 2$  by

$$g(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$$

$g$  is a Boolean algebra homomorphism since:

- (1) if  $g(x \wedge y) = 1$ , then  $x \wedge y \in U$ , implies  $x, y \in U$ ,

and so  $g(x) = 1, g(y) = 1$ , thus  $g(x \wedge y) = g(x) \wedge g(y)$ .

(b) If  $g(x \wedge y) = 0$ , then  $x \wedge y \notin U$ . Hence  $x \notin U$  or  $y \notin U$ , and so  $g(x) = 0$  or  $g(y) = 0$ , that is,  $g(x \wedge y) = g(x) \wedge g(y)$ .

(2) (a) If  $g(x \vee y) = 1$ , then  $x \vee y \in U$ . Hence  $x \in U$  or  $y \in U$ , since if not, then  $x^*, y^* \in U^{(\dagger)}$ , and so  $x^* \wedge y^* \in U$ . But we have  $(x^* \wedge y^*) = (x \vee y)^*$ , and  $(x \vee y)^*$  cannot be an element of  $U$ , because if it is, then so is  $(x \wedge y) \wedge (x \vee y)^*$ , and in this case  $U$  will not be proper. So we must have  $x \in U$  or  $y \in U$  for otherwise we get a contradiction. Hence  $g(x) = 1$  or  $g(y) = 1$ , that is,  $g(x \vee y) = g(x) \vee g(y)$ .

(b) If  $g(x \vee y) = 0$ , then  $x \vee y \notin U$ , implies  $x, y \notin U$  and so  $g(x) = g(y) = 0$ , and we get  $g(x \vee y) = g(x) \vee g(y)$ .

(3) (a) if  $g(x^*) = 1$ , then  $x^* \in U$ , implies  $x \notin U$ , and so  $g(x) = 0$ , so we have  $g(x^*) = 1 = 0^* = (g(x))^*$ .

(b) if  $g(x^*) = 0$ , then  $x^* \notin U$ , and so  $x \in U^{(\dagger\dagger)}$ , that is,  $g(x) = 1$  and we have  $g(x^*) = 0 = 1^* = (g(x))^*$ .

So we have proved that  $g$  is a Boolean algebra homomorphism.

<sup>†</sup> See [B.&S.], lemma 1.3.1.

<sup>††</sup> Ibid.



Now consider  $g \circ f'$ , where  $f'$  is the Boolean algebra homomorphism, from  $L$  to  $B$ , induced by  $f$ . (See lemma 1.5.1.)  $g \circ f'$  is a Boolean algebra homomorphism, since it is a composition of Boolean algebra homomorphisms.

Define  $h: F \rightarrow 2$ , by  $h(Y) = (g \circ f')(|Y|)$  for all  $Y \in F$ .

We remark that  $h$  is a homomorphism from  $F$  to  $2$ . We demonstrate only one of the four conditions of the definition of a homomorphism, i.e., condition (4).

$$\begin{aligned} h(Y \supset Z) &= (g \circ f')(|Y \supset Z|) \\ &= (g \circ f')(|Y| \supset |Z|) \\ &= (g \circ f')(|Y| * \vee |Z|) \\ &= ((g \circ f')(|Y|)) * \vee (g \circ f')(|Z|) \\ &= (h(Y)) * \vee (h(Z)) \\ &= h(Y) \supset h(Z), \text{ as desired.} \end{aligned}$$

Finally, we get back to our non-valid formula  $X$ , and we notice that:

$$h(X) = (g \circ f')(|X|) = g(f'(|X|)) = 0$$

as  $f'(|X|) = f(X) \notin U$ , that is,  $2 \not\models_h X$ .

So we have shown that if  $X$  is a non valid formula, then there exists a homomorphism  $f$  from  $F$  to  $2$  such that  $2 \not\models_f X$ , which, using the law of contraposition, completes the "if" part of the theorem.

## 1.6 The Intuitionistic Propositional Calculus (IPC).

We have seen that in the case of CPC, there exists a truth-table system with finitely many truth values, such that under it the completeness theorem holds. Is this also true in the case of IPC? This is the question which we shall answer in the next section. But first, we give a formalization of IPC which has been taken from [C].

As axioms, we take all formulae of one of the forms

$$\text{IPC1. } X \supset (Y \supset X)$$

$$\text{IPC2. } (Z \supset (X \supset Y)) \supset ((Z \supset X) \supset (Z \supset Y))$$

$$\text{IPC3. } X \wedge Y \supset X$$

$$\text{IPC4. } X \wedge Y \supset Y$$

$$\text{IPC5. } (X \supset (Y \supset (X \wedge Y)))$$

$$\text{IPC6. } X \supset X \vee Y$$

$$\text{IPC7. } Y \supset X \vee Y$$

$$\text{IPC8. } (Z \supset X) \supset ((Y \supset X) \supset ((Z \vee Y) \supset X))$$

$$\text{IPC9. } (X \supset \sim X) \supset \sim X$$

$$\text{IPC10. } \sim X \supset (X \supset Y)$$

The only rule of inference that we have is modus ponens.

A proof of a formula  $X$  is a finite sequence  $X_1, \dots, X_n$  of formulae such that  $X_n$  is  $X$  and for each  $k \leq n$ ,  $X_k$  is either an axiom or for some  $i, j < k$  is an immediate

consequence of  $X_1$  and  $X_2$  according to the rule modus ponens. A formula  $X$  is said to be provable if there exists a proof of it. If  $X$  is provable, we write  $\vdash X$ .

### 1.7 Godel's Counterexample.

This section is devoted to give a complete exposition of Godel's counterexample to the existence of a truth-table method with finitely many truth values such that under it the completeness theorem holds. (An outline of the counterexample may be found in [C].)

We start by giving the definition of a valid formula. Let  $A$  be any initial segment of the natural numbers,  $0, 1, \dots, v$ , (say). An assignment is a function  $f$  from the set of all atomic formulae  $AF$  to  $A$ . Given an assignment  $f$ , we extend it to the set of all formulae by the following recursive definition. For any  $X, Y$  if  $f(X)$  and  $f(Y)$  have already been defined, then

$$(1) \quad f(X \wedge Y) = \max(f(X), f(Y)).$$

$$(2) \quad f(X \vee Y) = \min(f(X), f(Y)).$$

$$(3) \quad f(\sim X) = \begin{cases} 0 & f(X) = v \\ v & \text{otherwise} \end{cases}$$

$$(4) \quad f(X \supset Y) = \begin{cases} 0 & f(X) \geq f(Y) \\ f(Y) & \text{otherwise} \end{cases}$$

$X$  is said to be valid in  $(A, f)$  if  $f(X) = 0$  and we write  $A \models_f X$ .  $X$  is called valid, in symbols  $\models X$ , if it is valid in every  $(A, f)$ .

Lemma 1.7.1. According to the above definition of validity the theorems of IPC are valid.

Proof. The proof is by induction on the length of proofs. We show that each axiom is valid and that if  $X$  and  $X \supset Y$  are valid then so is  $Y$ . It will follow that a proof consists of a sequence of valid formulae and, in particular, that every theorem is valid.

We first show that any instance of the axiom schema

IPC2.  $(Z \supset (X \supset Y)) \supset ((Z \supset X) \supset (Z \supset Y))$

is valid. To do so, assume that it is not valid, hence we must find  $(A, f)$  such that  $A \not\models_f (Z \supset (X \supset Y) \supset ((Z \supset X) \supset (Z \supset Y)))$  which means that  $f((Z \supset X) \supset (Z \supset Y)) > 0$ , but that is true only if  $f(Z \supset Y) > 0$ ,  $f(Y) > f(Z)$ . We have two cases:

(1)  $f(X) \leq f(Z)$ , that is,  $f(X) < f(Y)$ , and we have

$f(X \supset Y) \equiv f(Y)$ , thus  $f(Z \supset (X \supset Y)) \equiv f(Y) =$

$f((Z \supset X) \supset (Z \supset Y))$ . Hence  $f((Z \supset (X \supset Y)) \supset ((Z \supset X) \supset (Z \supset Y))) = 0$  and the axiom is valid in  $(A, f)$ .

(2)  $f(Z) < f(X)$ , we must have  $f(X) < f(Y)$  for otherwise  $f((Z \supset X) \supset (Z \supset Y)) \equiv 0$ , and the axiom will be valid in  $(A, f)$ . But in this case, we have again

$f((Z \supset (X \supset Y)) = f(Y) \sim f((Z \supset X) \supset (Z \supset Y))$  and the axiom is valid in  $(A, f)$ .

So our assumption led into a contradiction, and the axiom is valid for any  $Z, X$  and  $Y$ . The other axiom schemas can be checked in a similar way.

Now suppose that  $X$  and  $X \supset Y$  are valid. It follows from the validity of  $X$  that in every  $(A, f)$ ,  $f(X) = 0$ . And so  $f(Y) = 0$  for otherwise  $f(X \supset Y)$  will be equal to  $f(Y) > 0$  which contradicts the validity of  $(X \supset Y)$ . This proves the validity of  $Y$  and completes the proof of the lemma.

Now we prove the main result of this section.

Theorem 1.7.1. (Godel's Counterexample) There is no truth-table system for IPC with finitely many truth values such that under it the completeness theorem holds.

Proof. Consider the formula

$$(P_1 \supset P_2) \vee (P_1 \supset P_3) \vee \dots \vee (P_1 \supset P_n) \\ \vee (P_2 \supset P_3) \vee (P_2 \supset P_4) \vee \dots \vee (P_2 \supset P_n) \vee \dots \vee (P_{n-1} \supset P_n)$$

where  $n$  is arbitrary. The above formula is not a theorem of IPC as it is not valid in  $\langle \{1, 2, \dots, n\}, f \rangle$  where  $f(P_i) = i$ , since we have  $f(P_i \supset P_j) = j$  for  $i < j$ , and the formula takes the value 2. However, it becomes valid

upon identifying any two of its propositional variables by substituting one of the variables everywhere for the other. Since in this case we get  $(P_i \supset P_i)$  as a subformula, for some  $i \leq n$ , and in any  $(A, f)$ ;  $f(P_i \supset P_i) = 0$ , and so the whole formula takes the value zero.

Now assume that there is a truth-table system with finitely many truth values,  $m$  (say), such that the condition of the theorem holds, and consider the above formula with  $n$  equals  $m + 1$ , hence any assignment of  $P_1, \dots, P_{m+1}$  will take at least two of them to one truth value, and so the formula will be a tautology with respect to this system of truth-tables, and as a result of the completeness theorem it must be provable in IPC, which latter is not the case as we have shown earlier. This completes the proof of the theorem.

### 1.8 Algebraic Semantics.

It was after Godel had come up with his counterexample (Th. 1.7.1) that a complete semantics of IPC was developed, namely the algebraic semantics<sup>†</sup>. Here we state it. In the next section, we show how the work of section (1.7) can be done with respect to this semantics.

Definition 1.8.1: A Pseudo-Boolean algebra (PBA) is a pair

<sup>†</sup> Kripke made up another semantics for IPC, and it has been proved (in [F]) that the two semantics are equivalent.

$\langle B, \leq \rangle$  where  $B$  is a non-empty set and  $\leq$  is a partial ordering relation on  $B$  such that the following conditions are satisfied:

- (1) For any two elements  $a$  and  $b$  of  $B$ , the least upper bound  $(a \vee b)$ , and the greatest lower bound  $(a \wedge b)$ , exist.
- (2) For any two elements  $a$  and  $b$  of  $B$ , the pseudo complement of  $a$  relative  $b$ ,  $(a \Rightarrow b)$ , defined to be the greatest  $x \in B$  such that  $a \wedge x \leq b$ , exists.
- (3) A least element  $0$  exists.

Let  $-a$  be  $a \Rightarrow 0$  and  $1$  be  $-0$ .

We shall to a PBA  $\langle B, \leq \rangle$ , by its underlying set  $B$  when no confusion can result. A thorough treatment of Pseudo Boolean algebras may be found in [R.&S.].

Definition 1.8.2?  $h$  is called a homomorphism (from the set  $F$  of formulae to the PBA  $\langle B, \leq \rangle$ ) if  $h: F \rightarrow B$  and

- (1)  $h(X \wedge Y) = h(X) \wedge h(Y)$
- (2)  $h(X \vee Y) = h(X) \vee h(Y)$
- (3)  $h(\sim X) = -h(X)$
- (4)  $h(X \supset Y) = h(X) \Rightarrow h(Y)$

If  $\langle B, \leq \rangle$  is a PBA and  $h$  is a homomorphism, the triple  $\langle B, \leq, h \rangle$  is called an algebraic model for the IPC. A formula  $X$  is called valid in the model  $\langle B, \leq, h \rangle$  if

$h(X) = 1$ .  $X$  is called valid if  $X$  is valid in every model.

### 1.9. The Proof of Godel's Counterexample Using the Algebraic Semantics.

First we notice that lemma 1.7.1 remains true under the algebraic semantics since the axioms are valid and if  $X \supset Y$  and  $X$  are valid then so is  $Y$ . We show that any instance of the axiom schema

$$(Z \supset (X \supset Y)) \supset ((Z \supset X) \supset (Z \supset Y))$$

is valid. To do this, it is sufficient to prove that in any model  $\langle B, \leq, h \rangle$  we have

$$(h(Z) \Rightarrow (h(X) \Rightarrow h(Y))) \leq ((h(Z) \Rightarrow h(X)) \Rightarrow (h(Z) \Rightarrow h(Y))) \quad [1]$$

But that is true since

$$\begin{aligned} & h(Z) \wedge ((h(Z) \Rightarrow h(X)) \wedge (h(Z) \Rightarrow (h(X) \Rightarrow h(Y)))) \\ &= (h(Z) \wedge (h(Z) \Rightarrow h(X))) \wedge (h(Z) \wedge (h(Z) \Rightarrow (h(X) \Rightarrow h(Y)))) \\ &\leq h(X) \wedge (h(X) \Rightarrow h(Y)) \\ &\leq h(Y) \end{aligned}$$

and this proves [1]. The other axiom schemas can be checked in a similar way.

Now suppose that  $X \supset Y$  and  $X$  are valid. It follows that in every model  $\langle B, \leq, h \rangle$  we have



$$h(X \supset Y) = h(X) = h(Y) = 1, \text{ and}$$

$$h(X) = 1$$

Hence  $h(Y) = 1$  for otherwise  $h(X \supset Y) = h(Y) \neq 1$ . Thus we have proved lemma 1.7.1 using the algebraic semantics.

To do the same for theorem 1.7.1 all we have to prove is that the formula

$$(P_1 \supset P_2) \vee (P_1 \supset P_3) \vee \dots \vee (P_1 \supset P_n)$$

$$\vee (P_2 \supset P_3) \vee (P_2 \supset P_4) \vee \dots \vee (P_2 \supset P_n) \vee \dots \vee (P_{n-1} \supset P_n)$$

is not valid, for arbitrary  $n$ , and it becomes valid upon identifying any two of its propositional variables. The second part is clear since in this case we get  $P_i \subset P_i$ ,  $i \leq n$  as a subformula. But  $h(P_i \subset P_i) = h(P_i) \supset h(P_i) = 1$  in any model  $\langle B, \leq, h \rangle$ , and so the whole formula becomes valid. To prove the first part, consider the PBA  $\langle B, \leq, h \rangle$  where  $B = \{1, 2, \dots, n+1\}$ , and  $\leq$  is the usual ordering relation on the natural numbers restricted on  $B$ . (The greatest lower bound of any two elements  $i$  and  $j$  of  $B$  is defined to be their minimum and the least upper bound of  $i$  and  $j$  is their maximum.) Finally,  $h$  will be any homomorphism whose value at  $P_i$  is equal to  $1 - i$ ,  $1 \leq i \leq n$ . Such a homomorphism exists.

Now let  $i < j < n$ , we have

$$\begin{aligned} h(P_i \supset P_j) &= h(P_i) \Rightarrow (P_j) \\ &= (n+1-i) \Rightarrow (n+1-j) \\ &= n+1-j \end{aligned}$$

so we have

$$\begin{aligned} &h((P_1 \supset P_2) \vee \dots \vee (P_1 \supset P_n) \vee (P_2 \supset P_3) \vee \dots \vee (P_2 \supset P_n) \vee \dots \vee (P_{n-1} \supset P_n)) \\ &= h(P_1 \supset P_2) \vee \dots \vee h(P_1 \supset P_n) \vee h(P_2 \supset P_3) \vee \dots \vee h(P_2 \supset P_n) \vee \dots \vee h(P_{n-1} \supset P_n) \\ &= (n-1) \vee (n-2) \vee \dots \vee 1 \\ &\quad \vee (n-2) \vee (n-3) \vee \dots \vee 1 \vee \dots \vee 1 \\ &= n-1 \neq n+1 = 1_B, \end{aligned}$$

where  $1_B$  is the maximum element of the PBA  $\langle B, \leq \rangle$ .

This completes the proof of theorem 1.7.1 using the algebraic semantics.

#### 1.10. Infinite Truth-Table System for IPC.

We have seen that there is no truth-table system for IPC with finitely many truth values such that under it the completeness theorem holds. However, there are ones with infinitely many truth values. We shall illustrate them here. First this definition.

Definition 1.10.1: A subset  $A$  of a topological space  $X$  is said to be dense-in-itself provided

$$x \in C(A - \{x\}) \text{ for every } x \in A$$

where  $C(A - \{x\})$  is the closure of  $A - \{x\}$ .

Lemma 1.10.1: Every non-empty dense-in-itself metric space is infinite.

Proof. Let  $X$  be a non-empty dense-in-itself metric space. It follows that  $x \in C(X - \{x\})$  for every  $x \in X$ , hence  $\{x\}$  is not open for any  $x \in X$ . But any metric space is a  $T_1$ -space, and so we conclude that the interior  $A^\circ$  of any finite subset  $A$  of  $X$  is the empty set for otherwise  $\{x\}$  would be open for every  $x \in A^\circ$ . It follows that for every finite subset  $A$  of  $X$  we have

$$X = C(X - A)$$

which implies that  $X - A$  is not empty. Since this is true for any finite subset, we conclude that  $X$  is infinite.

Now we construct the Lindenbaum algebra of the intuitionistic calculus by the same method which we have used in the case of the classical propositional calculus. We have this theorem.

Theorem 1.10.1. For every formula  $X \in F$  (the set of all formulas of the intuitionistic propositional calculus) the following conditions are equivalent:

- (1)  $X$  is derivable
- (2)  $X$  is valid
- (3)  $X$  is valid in every Pseudo-Boolean algebra of open subsets of a topological space  $X$ .
- (4)  $X$  is valid in  $\langle L, \leq, h \rangle$ , where  $L$  is the Lindenbaum algebra of PPC and  $h$  is the homomorphism which assigns to every formula the equivalence class generated by it.
- (5)  $X$  is valid in every finite Pseudo-Boolean algebra.
- (6)  $X$  is valid in every Pseudo-Boolean algebra with at most  $2^{2^\gamma}$  elements where  $\gamma$  is the number of all subformulas of the formula  $X$ .
- (7)  $X$  is valid in the Pseudo-Boolean algebra of all open subsets of a non-empty dense-in-itself metric space.

Proof. See [R.&S.] .

It follows from (7) that we have a truth-table system for every non-empty dense-in-itself metric space  $D$ , with the elements of the Pseudo-Boolean algebra  $G(D)$  of the open subsets of  $D$  as truth values (there are an infinite number of them). Each truth-table has as many lines as the number of all possible homomorphisms from  $F$  to  $G(D)$ . As an example, we can take  $D$  to be equal to  $Q$ , the set of all rational numbers where the distance function  $d$  is defined by

$$d(\gamma_1, \gamma_2) = |\gamma_1 - \gamma_2| \quad \text{for all } \gamma_1, \gamma_2 \in Q$$

$(Q, d)$  is a non-empty dense-in-itself metric space.  $G(Q)$  is uncountable, i.e. we have an uncountable number of truth values.

It follows from (4) that we have a "degenerate" truth-table system with countably many truth values, namely the elements of the Lindenbaum algebra  $L$  (It is easy to prove that  $L$  is countable as a consequence of the fact that our alphabet is countable). It is sufficient for deciding the validity of any formula to check only the homomorphisms which assign to every formula the equivalence class generated by it. So, every truth-table reduces to one line only and that is why we call this system of truth-tables degenerate.

## CHAPTER II

### THE SEARCH FOR COUNTEREXAMPLES

#### 2.1 Introduction.

By a counterexample of a formula  $X$  in a logical system  $LS$ , we shall understand a model structure  $(A, f)$  of  $LS$  such that  $X$  is falsifiable in  $(A, f)$ . In this chapter, we shall develop a method of search for counterexamples to formulae of the classical propositional calculus. It will be shown how this method can be extended to the classical predicate calculus. We shall wind up by giving a method for the intuitionistic propositional calculus.

Condition (4) of Theorem 1.10.1 provides us with a counterexample to any non-valid formula of the intuitionistic propositional calculus. In the case of classical propositional and predicate calculi, analogous theorems of Theorem 1.10.1 can be proved<sup>†</sup>, and they also give us counterexamples for every non-valid formula of the classical propositional and predicate calculi. However, that does not affect the importance of developing systematic methods of search for counterexamples since, as we shall see later, these methods enable us to construct finite counterexamples to any non-valid formula in the classical and intuitionistic

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<sup>†</sup> (Th. VII, 2.1 & Th. VIII, 6.1, in [R.&S.1]).

propositional calculi. In the case of the classical predicate calculus, we shall be able either to prove the non-existence of a finite counterexample to a non-valid formula  $X$  and build an infinite one or to construct a finite counterexample.

Remark 2.1.1. A finite counterexample is a counterexample such that its underlying set is finite and an infinite counterexample is a counterexample such that its underlying set is infinite.

Remark 2.1.2. The existence of a finite counterexample to every non-valid formula is guaranteed by Theorem 1.5.1 in the case of the classical propositional calculus and by Theorem 1.10.1 "condition 5" in the case of the intuitionistic propositional calculus.

One might notice that a counterexample to every non-valid formula in the classical propositional calculus can be read off its truth-table, but again, this does not degrade the systematic method which we shall give for two reasons: - first, it is not easy to write the truth-tables for complicated formulae, and, secondly, our method can be extended to the classical predicate calculus, whereas there is no truth-table system for it.

The method which we shall give for the classical propositional calculus and its extension to the classical predicate calculus is due to Kleene (See [K]). It depends

very heavily on a slightly modified version of the Gentzen system which we expose in section 2.2. The original version may be found in [Sz].

## 2.2. The Gentzen System of the Classical Propositional Calculus.

We shall have the same alphabet as in section 1.1. We shall also have the same definition of a formula.

A sequent is an expression of the form

$$X_1, X_2, \dots, X_n \rightarrow Y_1, Y_2, \dots, Y_m$$

where  $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_m$  may represent any formula whatever. (The  $\rightarrow$ , like commas, is an auxiliary symbol and not a logical symbol.)

We call the formulae  $X_1, X_2, \dots, X_n$  the antecedent, and the formulae  $Y_1, Y_2, \dots, Y_m$  the succedent of the sequent. We shall also allow empty antecedent (empty succedent) if the succedent (antecedent) is not empty.

The sequent  $X_1, X_2, \dots, X_n \rightarrow Y_1, Y_2, \dots, Y_m$  is said to be true if and only if the formula

$$(X_1 \wedge X_2 \wedge \dots \wedge X_n) \supset (Y_1 \vee Y_2 \vee \dots \vee Y_m)$$

is true. If the antecedent is empty, the sequent reduces to the formula

$$Y_1 \vee Y_2 \vee \dots \vee Y_m$$



If the succedent is empty, the sequent means the same as the formula

$$\sim (X_1 \wedge X_2 \wedge \dots \wedge X_n)$$

An inference figure may be written in the following way:

$$\frac{\Gamma_1 \rightarrow \Theta_1, \Gamma_2 \rightarrow \Theta_2, \dots, \Gamma_n \rightarrow \Theta_n}{\Gamma \rightarrow \Theta}$$

where  $\Gamma_1, \Theta_1, \dots, \Gamma_n, \Theta_n, \Gamma, \Theta$  are sequences of formulae (possibly empty) separated by commas, in other words

$\Gamma_1 \rightarrow \Theta_1, \dots, \Gamma_n \rightarrow \Theta_n, \Gamma \rightarrow \Theta$  are sequents.  $\Gamma_1 \rightarrow \Theta_1, \dots, \Gamma_n \rightarrow \Theta_n$  are called the upper sequents and  $\Gamma \rightarrow \Theta$  the lower sequent of the inference figure.

A tree is a number of sequents (at least one) which combine to form inference figures in such a way that each sequent is the upper sequent of exactly one inference figure except one which we call the endsequent.

The sequents of a tree that are not lower sequents of an inference figure are called initial sequents of the tree.

A proof is a finite tree in which every sequent is a lower sequent of at most one inference figure, the system of inference is noncircular, i.e., there is in the proof no sequence whose last member is again succeeded by its first member, and the antecedent of the endsequent is empty and.

its succedent contains exactly one formula. Each initial sequent of the proof is of the form

$$X \rightarrow X$$

where  $X$  may be any arbitrary formula, and each inference figure of the proof results from one of the schemata below by replacing  $X, Y, Z, W$  by arbitrary formula and  $\Gamma_1, \theta_1, \Gamma_2, \theta_2$  by arbitrary sequences of formulae (possibly empty) separated by commas.

### The Inference Schemata.

(a) Schemata for structural inference figures.

#### Thinning

in the antecedent

$$\frac{\Gamma_1 \rightarrow \theta_1}{X, \Gamma_1 \rightarrow \theta_1}$$

in the succedent

$$\frac{\Gamma_1 \rightarrow \theta_1}{\Gamma_1 \rightarrow \theta_1, X}$$

#### Contraction

in the antecedent

$$\frac{X, X, \Gamma_1 \rightarrow \theta_1}{X, \Gamma_1 \rightarrow \theta_1}$$

in the succedent

$$\frac{\Gamma_1 \rightarrow \theta_1, X, X}{\Gamma_1 \rightarrow \theta_1, X}$$

#### Interchange

in the antecedent

$$\frac{\Gamma_1, X, Y, \Gamma_2 \rightarrow \theta}{\Gamma_1, Y, X, \Gamma_2 \rightarrow \theta}$$

in the succedent

$$\frac{\Gamma_1 \rightarrow \theta_1, X, Y, \theta_2}{\Gamma_1 \rightarrow \theta_1, Y, X, \theta_2}$$

Cut

$$\frac{\Gamma_1 \rightarrow \theta_1, X \quad X, \Gamma_2 \rightarrow \theta_2}{\Gamma_1, \Gamma_2 \rightarrow \theta_1, \theta_2}$$

(b) Schemata for operational inference figures.

$$\frac{X, \Gamma_1 \rightarrow \theta_1, Y}{\Gamma_1 \rightarrow \theta_1, X \supset Y} \supset \quad \frac{\Gamma_1 \rightarrow \theta_1, X \quad Y, \Gamma_1 \rightarrow \theta_1}{X \supset Y, \Gamma_1 \rightarrow \theta_1} \supset \rightarrow$$

$$\frac{\Gamma_1 \rightarrow \theta_1, X \quad \Gamma_1 \rightarrow \theta_1, Y}{\Gamma_1 \rightarrow \theta_1, X \wedge Y} \wedge \rightarrow \quad \frac{X, Y, \Gamma_1 \rightarrow \theta_1}{X \wedge Y, \Gamma_1 \rightarrow \theta_1} \wedge \rightarrow$$

$$\frac{\Gamma_1 \rightarrow \theta_1, X, Y}{\Gamma_1 \rightarrow \theta_1, X \vee Y} \vee \rightarrow \quad \frac{X, \Gamma_1 \rightarrow \theta_1 \quad Y, \Gamma_1 \rightarrow \theta_1}{X \vee Y, \Gamma_1 \rightarrow \theta_1} \vee \rightarrow$$

$$\frac{X, \Gamma_1 \rightarrow \theta_1}{\Gamma_1 \rightarrow \theta_1, \sim X} \sim \rightarrow \quad \frac{\Gamma_1 \rightarrow \theta_1, X}{\sim X, \Gamma_1 \rightarrow \theta_1} \sim \rightarrow$$

A proof of a formula  $X$  is a proof such that  $X$  is the unique formula in the succedent of the endsequent of the proof.  $X$  is provable if there is a proof for it.

It can be proved that for every provable formula  $X$ , we can find a proof for it in which the inference figure called "cut" does not occur. But that will be out of the scope of this thesis.

It can also be proved, (we shall not give the proof here), that the Gentzen system of the classical propositional calculus is equivalent to the Hilbert system in

the sense that both determine the same class of provable formulae.

### 2.3. Systematic Method of Search for Counterexamples to Formulae in the Classical Propositional Calculus.

Here we develop a systematic method which, as we have mentioned before, provides us with a finite counterexample to every non-valid formula in the classical propositional calculus. We start with the following lemma.

Lemma 2.3.1. For each of the operational inference figure schemata, the lower sequent is falsifiable in a model  $(B, f)$  if and only if the upper sequent, or at least one of the two upper sequents, is falsifiable in  $(B, f)$ .

Proof. Clear.

Now let  $X$  be any formula in the classical propositional calculus, consider the following sequent:

$$\rightarrow X$$

We apply the operational inference figures upward to form a tree. Define an "upward" path in a tree to be a sequence of sequents whose first sequent is the end sequent and whose last sequent, if any, is a sequent from which no further step upward can be made using the operational inference figure or a sequent of the form

$$Y, \Gamma \rightarrow \Theta, Y$$

and of which each sequent except the last (if it exists) is a lower sequent of an inference figure whose upper sequent is the next sequent in the path.

A path is said to be closed if its last sequent has the form

$$Y, \Gamma \rightarrow \perp, Y$$

and since any sequent of the above form is not falsifiable, we do not get a counterexample using a closed path, but we get a counterexample to  $X$  for every non-closed path, namely, the model structure which falsifies the last sequent of the path (See Lemma 2.3.1).

We notice that in the case of the classical propositional calculus, all paths are finite, but in the case of the classical predicate calculus, we shall have paths which can be pursued ad infinitum.

If all the paths in the tree for  $X$  (i.e., the tree whose endsequent is  $\rightarrow X$ ) are closed, we shall not get any counterexamples for  $X$ . But in this case  $X$  is provable since we can get a proof for it by applying the thinning inference figure upward enough times, and so  $X$  is valid (by the completeness theorem of the classical propositional calculus), thus we have a method which gives us at least one counterexample to every non-valid formula in the classical propositional calculus.

Example 1. Here we apply the previous method to search for a counterexample to the formula

$$(P_1 \supset (P_2 \supset P_3)) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$$

Consider the tree

		X		X		X
$P_2 \supset P_3, P_1$	$P_2, P_2 \supset P_3, P_1$	$P_2 \supset P_3, P_2, P_1$	$P_2 \supset P_3, P_2, P_1$	$P_2 \supset P_3, P_2, P_1$	$P_2 \supset P_3, P_2, P_1$	$P_2 \supset P_3, P_2, P_1$
$\rightarrow P_2 \supset P_3, P_1$	$\rightarrow P_2 \supset P_3, P_2, P_1$	$\rightarrow P_2 \supset P_3, P_2, P_1$	$\rightarrow P_2 \supset P_3, P_2, P_1$	$\rightarrow P_2 \supset P_3, P_2, P_1$	$\rightarrow P_2 \supset P_3, P_2, P_1$	$\rightarrow P_2 \supset P_3, P_2, P_1$
$P_1 \supset P_2 \supset P_3, P_1$	$P_1 \supset P_2 \supset P_3, P_1$	$P_1 \supset P_2 \supset P_3, P_1$	$P_1 \supset P_2 \supset P_3, P_1$	$P_1 \supset P_2 \supset P_3, P_1$	$P_1 \supset P_2 \supset P_3, P_1$	$P_1 \supset P_2 \supset P_3, P_1$
$\rightarrow (P_1 \supset P_2) \supset (P_2 \supset P_3), P_1$	$\rightarrow (P_1 \supset P_2) \supset (P_2 \supset P_3), P_1$	$\rightarrow (P_1 \supset P_2) \supset (P_2 \supset P_3), P_1$	$\rightarrow (P_1 \supset P_2) \supset (P_2 \supset P_3), P_1$	$\rightarrow (P_1 \supset P_2) \supset (P_2 \supset P_3), P_1$	$\rightarrow (P_1 \supset P_2) \supset (P_2 \supset P_3), P_1$	$\rightarrow (P_1 \supset P_2) \supset (P_2 \supset P_3), P_1$
$P_1 \supset (P_2 \supset P_3) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$	$P_1 \supset (P_2 \supset P_3) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$	$P_1 \supset (P_2 \supset P_3) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$	$P_1 \supset (P_2 \supset P_3) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$	$P_1 \supset (P_2 \supset P_3) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$	$P_1 \supset (P_2 \supset P_3) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$	$P_1 \supset (P_2 \supset P_3) \supset ((P_1 \supset P_2) \supset (P_2 \supset P_3))$

Hence we have  $(Z, f)$  as a counterexample of the above formula where  $f(P_2) = 1$  and  $f(P_3) = f(P_1) = 0$ , and  $f(P_1)$  is arbitrary if  $i \neq 1, 2, 3$ .

Note: By X we mean that the path is closed.

Example 2. In this example, we discuss the existence of a counterexample to the formula

$$(P_1 \vee P_2) \supset (P_1 \wedge P_2)$$

Consider the tree

$$\begin{array}{c}
 \text{X} \qquad \qquad \qquad \text{X} \\
 \hline
 \begin{array}{cc}
 P_1 \rightarrow P_1 & P_1 \rightarrow P_2 \\
 P_2 \rightarrow P_1 & P_2 \rightarrow P_2
 \end{array} \\
 \hline
 \begin{array}{cc}
 P_1 \rightarrow P_1 \wedge P_2 & P_2 \rightarrow P_1 \wedge P_2
 \end{array} \\
 \hline
 P_1 \vee P_2 \rightarrow P_1 \wedge P_2 \\
 \hline
 \rightarrow (P_1 \vee P_2) \supset (P_1 \wedge P_2)
 \end{array}$$

So in this case, we have two counterexamples to the above formula:

1)  $(2, f_1)$  where  $f_1(P_1) = 1$  and  $f_1(P_2) = 0$ , and  $f_1(P_i)$  is arbitrary if  $i \neq 1, 2$ .

2)  $(2, f_2)$  where  $f_2(P_1) = 0$  and  $f_2(P_2) = 1$ , and  $f_2(P_i)$  is arbitrary if  $i \neq 1, 2$ .

Example 3. In this example, we consider the formula

$$(P_1 \supset P_2) \supset ((P_3 \supset P_1) \supset (P_3 \supset P_2))$$

To discuss the existence of a counterexample to the above formula, we examine the following tree.

Path (1)

X	X	X	X
$P_3 \rightarrow P_1, P_2, P_3$	$P_1, P_3 \rightarrow P_2, P_1$	$P_2, P_3 \rightarrow P_2, P_3$	$P_2, P_1, P_3 \rightarrow P_2$
$\rightarrow P_3 \supset P_2, P_1, P_3$	$P_1 \rightarrow P_3 \supset P_2, P_1$	$P_2 \rightarrow P_3 \supset P_2, P_3$	$P_2, P_1 \rightarrow P_3 \supset P_2$
$P_3 \supset P_1 \rightarrow P_3 \supset P_2, P_1$		$P_2, (P_3 \supset P_1) \rightarrow P_3 \supset P_2$	
$\rightarrow (P_3 \supset P_1) \supset (P_3 \supset P_2), P_1$		$P_2 \rightarrow (P_3 \supset P_1) \supset (P_3 \supset P_2)$	
$P_1 \supset P_2 \rightarrow (P_3 \supset P_1) \supset (P_3 \supset P_2)$			
$\rightarrow (P_1 \supset P_2) \supset ((P_3 \supset P_1) \supset (P_3 \supset P_2))$			

Here, all paths are closed, and so we do not get a counterexample, but the above tree can be completed to a proof of our formula. As an example, we apply the thinning inference figure upward twice on the last sequent of path (1) to get the sequent  $P_3 \rightarrow P_3$  as required in the definition of the proof. We can do the same in every other path, and so the above formula is provable, hence it is valid.

#### 2.4 The Classical Predicate Calculus and the Gentzen System.

In this section, we shall consider the alphabet which consists of a countably infinite set of variables  $v_1, v_2, \dots$ , a countably infinite set of predicate letters  $P_1, P_2, \dots$ , four logical connectives  $\wedge, \vee, \supset, \sim$ , two quantifier



symbols  $\forall, \exists$  and  $(, )$  as punctuation symbols. Each predicate letter  $P_n$  is associated with a non-negative integer  $d(n)$  called the degree of  $P_n$ .

An atomic formula is defined to be any expression of the form

$$P_n(v_{i_0}, \dots, v_{i_{d(n)}})$$

We now give the following recursive definition of a formula:

- (1) An atomic formula is a formula.
- (2) If  $X$  is a formula, so is  $\sim X$ .
- (3) If  $X$  and  $Y$  are formulas, so are  $(X \wedge Y)$ ,  $(X \vee Y)$ ,  $(X \supset Y)$ .
- (4) If  $X$  is a formula, so are  $(\exists v)X$  and  $(\forall v)X$ .

Again, we omit writing outer parentheses in a formula when no confusion can result.

An occurrence of a variable  $v$  in a formula  $X$  is said to be a free occurrence if it occurs in a subformula  $Y$  of  $X$  which is not contained as a subformula of any subformula of  $X$  which has one of the forms

$$\exists v Z, \forall v Z$$

otherwise it is said to be a bound occurrence of the variable.

$v$ . Any particular occurrence is either free or bound, but not both, however, a variable  $v$  can have both free and bound occurrence in the same formula. (The definition of immediate subformula follows the same pattern as that of the definition of section 1.3. We only replace the propositional variable in ISO by atomic formula and add the sentence which states that  $X$  is an immediate subformula of both  $\exists v X$  and  $\forall v X$ , the definition of a subformula is exactly the same as the definition in section 1.1.)

A model structure of the classical predicate calculus is a pair  $(\bar{A}, f)$  where  $\bar{A}$  is a relational structure  $(A, \{R_n : n \in \omega\})$ , where  $A$  is any non-empty set and for every  $n \in \omega$ ,  $R_n$  is  $d(n)$ -ary relation defined on  $A$ , where  $d(n)$  is the degree of the predicate  $P_n$ , and  $f$  is a function from the set of all variables to  $A$ .

We define the relation ' $f$  satisfies  $X$  in  $\bar{A}$ ' in symbols  $\bar{A} \models_f X$  recursively as follows:

- (1)  $\bar{A} \models_f P_n(v_{i_1}, \dots, v_{i_{d(n)}})$  if and only if  $\langle f(v_{i_1}), \dots, f(v_{i_{d(n)}}) \rangle \in R_n$ .
- (2)  $\bar{A} \models_f \sim X$  if and only if  $\bar{A} \not\models_f X$ .
- (3)  $\bar{A} \models_f X \wedge Y$  if and only if  $\bar{A} \models_f X$  and  $\bar{A} \models_f Y$ .
- (4)  $\bar{A} \models_f X \vee Y$  if and only if  $\bar{A} \models_f X$  or  $\bar{A} \models_f Y$ .

(5)  $\bar{A} \models_f X \supset Y$  if and only if  $\bar{A} \not\models_f X$  or  $\bar{A} \models_f Y$ .

(6)  $\bar{A} \models_f (\exists v_n) X$  if and only if for some  $a \in A$ ,  $\bar{A} \models_{f(n/a)} X$ .

(7)  $\bar{A} \models_f (\forall v_n) X$  if and only if for every  $a \in A$ ,

$$\bar{A} \models_{f(n/a)} X.$$

where  $f(n/a)$  is the function from the set of all variables  $V$  to  $A$  which assigns  $a$  to  $v_n$  and agrees with  $f$  on every other variable. We say that  $X$  is universally valid if it is valid in every model structure. We get the Gentzen system of the classical predicate calculus from that of the classical propositional calculus by adding the following operational inference figure schemata.

$$\frac{\Gamma \rightarrow \theta, X(b)}{\Gamma \rightarrow \theta, \forall v X(v)} \rightarrow \forall \quad \frac{X(b), \forall v X(v), \Gamma \rightarrow \theta}{\forall v X(v), \Gamma \rightarrow \theta} \forall \rightarrow$$

where  $b$  does not  
occur free in  
 $\Gamma \rightarrow \theta, \forall v X(v)$ .

$$\frac{\Gamma \rightarrow \theta, \exists v X(v), X(b)}{\Gamma \rightarrow \theta, \exists v X(v)} \rightarrow \exists \quad \frac{X(b), \Gamma \rightarrow \theta}{\exists v X(v), \Gamma \rightarrow \theta} \exists \rightarrow$$

where  $b$  does not occur  
free in  $\exists v X(v), \Gamma \rightarrow \theta$ .

Now we are able to extend the method of section 2.3 to the classical predicate calculus.

## 2.5. Systematic Method of Search for Counterexamples to Formulae in the Classical Predicate Calculus.

We notice that lemma 2.3.1 holds for inference figure schemata introduced in the previous section. Now let  $X$  be any formula and  $u_0, \dots, u_\ell$  be a list (possibly empty) of the free variables of  $X$ . Consider the tree which has  $\rightarrow X$  as the endsequent and is constructed according to the following pattern: We group the steps along any path in rounds; to explain how round  $d$ , say, in any particular path is carried out, assume that after  $d-1$  rounds have been completed we have a sequent of the form

$$X_1, \dots, X_n \rightarrow Y_1, \dots, Y_m$$

We apply on the above sequent the relevant rule of the ones written below with  $X_j$  replaced by  $X_1$ ; we do the same to the resulting sequent with  $X_j$  replaced by  $X_2$  and so on till we finish with  $X_n$ , then we do the same as above with  $Y_j$  (in the rules) replaced by each one in turn of  $Y_1, \dots, Y_m$ .

The rules are

- (1) If  $X_j(Y_j)$  is atomic, no step is performed.
- (2) If the outermost logical symbol of  $X_j(Y_j)$  is  $\wedge, \vee, \supset$  or  $\sim$  we apply the inference figure  $\wedge \rightarrow, \vee \rightarrow, \supset \rightarrow$  or  $\sim \rightarrow$  ( $\rightarrow \wedge, \rightarrow \vee, \rightarrow \supset$  or  $\rightarrow \sim$ ) upward, respectively with  $X_j(Y_j)$  being the only formula to be affected by the application of the inference figure.

(3) If the outermost logical symbol of  $X_j(Y_j)$  is  $\exists(V)$ , we apply the inference figure  $\exists \rightarrow (+V)$  upward with  $a_i$  as the  $b$ , if at this point we have done  $i - 1$  steps using  $\exists \rightarrow$  and  $+V$  with  $a_0, \dots, a_{i-1}$  as the  $b$  of the inference figure. (We call  $a_0, \dots, a_i$  the activated variables and in each application of  $\exists \rightarrow$  or  $+V$  we introduce a new activated variable.)

(4) If the outermost logical symbol of  $X_j(Y_j)$  is  $\forall(\exists)$ , we apply the inference figure  $\forall \rightarrow (+\exists)$  upward using as the  $b$  each one in turn of the variables  $u_0, \dots, u_p, a_0, \dots, a_i$  which has not previously served as the  $b$  of the inference figure  $\forall \rightarrow (+\exists)$  applied with respect to  $X_j(Y_j)$  where  $a_0, \dots, a_i$  are all the activated variables we have thus far. (By the application of  $\forall \rightarrow$  with respect to  $X_j$ , we mean that  $X_j$  has the form  $\forall v Z(v)$ , and we have a sequent of the form  $\Gamma, \forall v Z \rightarrow \Theta$ , and we apply  $\forall \rightarrow$  on it to get a sequent of the form  $\Gamma, Z(b), \forall v Z(v) \rightarrow \Theta$ .) If at this point we do not have any activated variable which has not previously served as the  $b$  of  $\forall \rightarrow (+\exists)$  we apply  $\forall \rightarrow (+\exists)$  with  $a_i$  as the  $b$ , if we already have  $i - 1$  activated variables, and in this case we consider  $a_i$  as an activated variable.

Now assume that the above tree has an unclosed path where  $a_0, \dots, a_r$  or  $a_0, a_1, a_2, \dots$  are the activated variables along the path, (we shall have infinitely

many variables if the path does not terminate). Let  $A$  be the set  $\{0, \dots, r+2+1\}$  or the set  $\{0, 1, 2, \dots\}$  respectively, and  $f$  be a function from the set of all variables  $V$  to  $A$  where  $f(u_i) = i$ ,  $f(a_i) = i+2+1$  and  $f$  is arbitrary elsewhere. Let  $\bar{A}$  be the relational structure whose underlying set is  $A$  and whose relations are defined in such a way that each atomic formula in the antecedent (succedent) of any sequent along the path becomes true (false) in the model structure  $(\bar{A}, f)$ . It is easy to see that such a model structure exists.

It can be proved (we shall only demonstrate that by giving some examples) that not only the atomic formulae, but also every formula in the antecedent (succedent) of any sequent along the path is true (false) in the above model structure, and so  $(\bar{A}, f)$  is a counterexample to  $X$ .

We can see from the above discussion that we shall not get a counterexample to  $X$  only if all the paths of the above tree are closed, but in this case the tree can be completed to a proof for  $X$  (See section 2.3) which means that  $X$  is universally valid (by the completeness theorem of the classical predicate calculus).

Example 1. In this example, we consider the formula

$$\sim((\exists v_1 (P_1(v_1) \supset P_2(v_1)) \supset \forall v_1 (P_1(v_1) \supset P_2(v_1))) \supset (\forall v_1 \exists v_2 P_3(v_1, v_2)))$$

To discuss the existence of counterexamples to the above formula (denote it by  $\sim X$ ), we examine the following trees.

(path 1)	(path 2)	(path 3)
$  \begin{array}{l}  P_1(a_0) \rightarrow P_1(a_1), P_2(a_0) \quad P_2(a_1), P_1(a_0) \rightarrow P_2(a_0) \\  \hline  \rightarrow P_1(a_1), P_1(a_0) \supset P_2(a_0) \quad P_2(a_1) \rightarrow P_1(a_0) \supset P_2(a_0) \\  \hline  P_1(a_1) \supset P_2(a_1) \rightarrow P_1(a_0) \supset P_2(a_0) \quad \dots \\  \hline  \exists v_1 (P_1(v_1) \supset P_2(v_1)) \rightarrow P_1(a_0) \supset P_2(a_0) \\  \hline  \exists v_1 (P_1(v_1) \supset P_2(v_1)) \rightarrow \forall v_1 (P_1(v_1) \supset P_2(v_1)) \\  \hline  \rightarrow \exists v_1 (P_1(v_1) \supset P_2(v_1)) \supset \forall v_1 (P_1(v_1) \supset P_2(v_1)) \\  \hline  (\exists v_1 (P_1(v_1) \supset P_2(v_1)) \supset \forall v_1 (P_1(v_1) \supset P_2(v_1))) \supset (\forall v_1 \exists v_2 P_3(v_1, v_2)) \rightarrow  \end{array}  $	$  \begin{array}{l}  P_3(a_0, a_1), P_3(a_1, a_2), P_3(a_2, a_3), \forall v_1 \exists v_2 P_3(v_1, v_2) \rightarrow \\  \hline  P_3(a_0, a_1), P_3(a_1, a_2), \exists v_2 P_3(a_2, v_2), \forall v_1 \exists v_2 P_3(v_1, v_2) \rightarrow \\  \hline  P_3(a_0, a_1), P_3(a_1, a_2), \forall v_1 \exists v_2 P_3(v_1, v_2) \rightarrow \\  \hline  P_3(a_0, a_1), \exists v_2 P_3(a_1, v_2), \forall v_1 \exists v_2 P_3(v_1, v_2) \rightarrow \\  \hline  P_3(a_0, a_1), \forall v_1 \exists v_2 P_3(v_1, v_2) \rightarrow \\  \hline  \exists v_2 P_3(a_0, v_2), \forall v_1 \exists v_2 P_3(v_1, v_2) \rightarrow \\  \hline  \forall v_1 \exists v_2 P_3(v_1, v_2) \rightarrow  \end{array}  $	

$\rightarrow \sim X$

From "path 1", we get the counterexample whose underlying set  $A$  is  $\{0,1,2\}$ , and whose  $f$  is defined in such a way that  $f(a_i) = i$ ,  $i = 1,2,3$ , where  $R_1$  can be any subset of  $A$  which contains zero and  $R_2$  any subset of  $A$  which does not contain zero. (Here, and in any other example, we are only interested in the definition of the correspondent relations of the predicate letters which occur in the path under consideration, since the other relation can be defined arbitrarily.

From "path 2", we get a counterexample similar to the one we got from "path 1".

From "path 3", we get two different counterexamples. The first one is  $(\bar{A}, f)$ , where  $A$  is the set  $\{0,1,2,\dots\}$ ,  $R_3$  is the set  $\{(i, i+1 ; i \in \omega)\}$ , and  $f$  is any function such that  $f(a_i) = i$ ,  $i = 0,1,2,\dots$ . For the second counterexample we have  $\{0\}$  as the underlying set, where  $f$  is any function which assigns to each  $a_i$  the value zero, that is the only functions we have from  $V$  to  $A$ , and  $R_3$  is the set  $\{(0,0)\}$ .



Example 2. Consider the following tree

X

$$\begin{array}{c}
 P_1(a_0), \forall v_1 P_1(v_1) \rightarrow P_1(a_0), \exists v_1 P_1(v_1) \\
 \hline
 P_1(a_0), \forall v_1 P_1(v_1) \rightarrow \exists v_1 P_1(v_1) \\
 \hline
 \forall v_1 P_1(v_1) \rightarrow \exists v_1 P_1(v_1) \\
 \hline
 \rightarrow \forall v_1 P_1(v_1) \supset \exists v_1 P_1(v_1) \quad |
 \end{array}$$

The unique path of the above tree is closed, but we can get a proof for the endsequent of the tree by applying the thinning inference figure upward twice on the sequent

$P_1(a_0), \forall v_1 P_1(v_1) \rightarrow P_1(a_0), \exists v_1 P_1(v_1)$ , to get the sequent  $P_1(a_0) \rightarrow P_1(a_0)$  as required in the definition of the proof, and so the endsequent is provable, which means that it is universally valid (by the completeness theorem of the classical predicate calculus).

Example 3. In the following diagram, each step represents one or more applications of antecedent of any sequent which we do not want to highlight. (The diagram is not of them are closed.)

X

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

$P(a_0, a_1), P(a_1, a_2), \Gamma \rightarrow P(a_0, a_1), P(a_0, a_0)$

$P(a_0, a_1), P(a_0, a_2), P(a_1, a_2), \Gamma \rightarrow P(a_0, a_0), P(a_1, a_1), P(a_2, a_2) \quad P(a_0, a_1), P(a_1, a_2), \Gamma \rightarrow P(a_0, a_1) \wedge P(a_1, a_2)$

$P(a_0, a_1) \wedge P(a_1, a_2) \supset P(a_0, a_2), P(a_0, a_1), P(a_1, a_2), \Gamma \rightarrow P(a_0, a_0), P(a_1, a_1), P(a_2, a_2)$

$\neg P(a_2, a_2), P(a_0, a_1) \wedge P(a_1, a_2) \supset P(a_0, a_2), P(a_0, a_1), P(a_1, a_2), \Gamma \rightarrow P(a_0, a_0), P(a_1, a_1)$

$P(a_0, a_1) \wedge P(a_1, a_2) \supset P(a_0, a_2), P(a_0, a_1), P(a_1, a_2), \forall v_1 (\neg P(v_1, v_1)), \Gamma \rightarrow P(a_0, a_0)$

$\forall v_3 (P(a_0, a_1) \wedge P(a_1, v_3) \supset P(a_0, v_3)), P(a_0, a_1), \exists v_2 P(a_1, v_2), \Gamma \rightarrow P(a_0, a_0), P(a_1, a_1)$

$\neg P(a_1, a_1), \forall v_3 (P(a_0, a_1) \wedge P(a_1, v_3) \supset P(a_0, v_3)), P(a_0, a_1), \forall v_1 \exists v_2 P(v_1, v_2), \Gamma \rightarrow P(a_0, a_0)$

$\forall v_2 \forall v_3 (P(a_0, v_2) \wedge P(v_2, v_3) \supset P(a_0, v_3)), P(a_0, a_1), \forall v_1 (\neg P(v_1, v_1)), \Gamma \rightarrow P(a_0, a_0)$

$\neg P(a_0, a_0), \forall v_2 \forall v_3 (P(a_0, v_2) \wedge P(v_2, v_3) \supset P(a_0, v_3)), \exists v_2 P(a_0, v_2), \forall v_1 (\neg P(v_1, v_1))$

$\forall v_1 (\neg P(v_1, v_1)), \forall v_1 \forall v_2 \forall v_3 (P(v_1, v_2) \wedge P(v_2, v_3) \supset P(v_1, v_3)), \forall v_1 \exists v_2 P(v_1, v_2) \rightarrow$

$\rightarrow \neg (\forall v_1 (\neg P(v_1, v_1)) \wedge \forall v_1 \forall v_2 \forall v_3 (P(v_1, v_2) \wedge P(v_2, v_3) \supset P(v_1, v_3)) \wedge \forall v_1 \exists v_2 P(v_1, v_2))$

the applications of inference figures. We shall denote by  $\Gamma$  all the formulae in the diagram. The diagram is not a complete tree since we omit infinitely many paths, however all

X

X

$\Gamma \vdash P(a_0, a_1), P(a_0, a_0), P(a_1, a_1), P(a_2, a_2) \quad P(a_0, a_1), P(a_1, a_2), \Gamma \vdash P(a_1, a_2), P(a_0, a_0), P(a_1, a_1), P(a_2, a_2)$

$\Gamma \vdash P(a_0, a_1) \wedge P(a_1, a_2), P(a_0, a_0), P(a_1, a_1), P(a_2, a_2)$

$P(a_0, a_0), P(a_1, a_1), P(a_2, a_2)$

$P(a_2), \Gamma \vdash P(a_0, a_0), P(a_1, a_1)$

$\neg P(v_1, v_1), \Gamma \vdash P(a_0, a_0), P(a_1, a_1)$

$\neg P(a_2), \Gamma \vdash P(a_0, a_0), P(a_1, a_1)$

$\neg P(v_1, v_2), \Gamma \vdash P(a_0, a_0)$

$\neg P(v_1, v_1), \Gamma \vdash P(a_0, a_0)$

$\neg P(a_0, v_2), \neg P(v_1, v_1), \Gamma \vdash$

$\neg P(v_1, v_2), \neg P(v_1, v_2) \vdash$

$\neg P(v_3) \wedge \neg P(v_2) \vdash$

To get a counterexample from the unique unclosed path of the diagram, which is the unique unclosed path of the complete tree, to the endformula, i.e., the unique formula in the endsequent, we have to have a model structure which falsifies the atomic formulae

$$P(a_0, a_0), P(a_1, a_1), \dots, P(a_n, a_n), \dots$$

and satisfies the atomic formulae

$$P(a_0, a_1), P(a_0, a_2), \dots, P(a_0, a_n), \dots$$

$$P(a_1, a_2), P(a_1, a_3), \dots, P(a_1, a_n), \dots$$

$$P(a_n, a_{n+1}), P(a_n, a_{n+2}), \dots$$

So we can take the set  $\{0, 1, 2, \dots\}$  as the  $A$  and define  $f(a_i)$  to be equal to  $i$  for each natural number  $i$ , and  $R$ , the correspondent relation to the predicate  $P$ , to be the usual ordering relation on the natural numbers, it is clear that  $(\bar{A}, f)$  satisfies the above conditions, and hence it is a counterexample to the endformula.

There is no finite counterexample to the end-formula since if there was one,  $((A, R), f)$ , say, we would have  $f(a_i) = f(a_j)$ , for some natural numbers  $i, j$  such that  $i < j$ , but the counterexample must falsify  $P(a_i, a_i)$  and satisfy  $P(a_i, a_j)$ , which means that  $(f(a_i), f(a_i)) \notin R$  and  $(f(a_i), f(a_j)) = (f(a_i), f(a_i)) \in R$ , and so the assumption of the existence of a finite counterexample led into a contradiction.

Remark. The existence of a systematic method which gives us a counterexample to every non-valid formula does not contradict the undecidability of the classical predicate calculus<sup>†</sup> since in some cases we can only get a counterexample from a path which does not terminate.

## 2.6. The Search for Counterexamples to Non-Valid Formulae

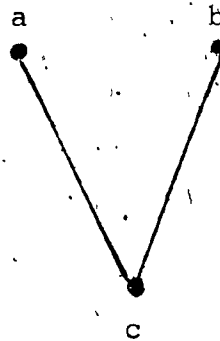
### In the Intuitionistic Propositional Calculus.

Any finite Pseudo-Boolean algebra  $B$  can be represented by a diagram in which the points indicate the elements of  $B$ , and for any  $a, b, c \in B$  the figures [1.1], [1.2] and [1.3] mean that  $a \leq b$ , the greatest lower bound of  $a$  and  $b$  is  $c$ , and the least upper bound of  $a$  and  $b$  is  $c$ , respectively.

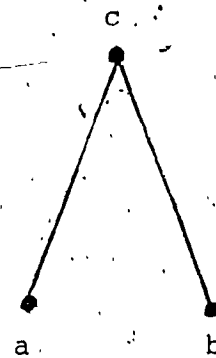
<sup>†</sup> A proof of the undecidability of the classical predicate calculus may be found in [C]<sub>1</sub>.



[1.1]



[1.2]



[1.3]

Now we give three lemmas which enable us to find out whether or not a given diagram represents a Pseudo-Boolean algebra.

Lemma 2.6.1. Every Pseudo-Boolean algebra is a distributive lattice.

Proof. See [R.&S.].

Lemma 2.6.2. Every finite distributive lattice is a Pseudo-Boolean algebra.

Proof. Let  $(B, \leq)$  be a finite distributive lattice,  $(B, \leq)$  has a zero element, namely,  $\bigwedge B$ .

Now, let  $b_1, b_2 \in B$ . Consider the set

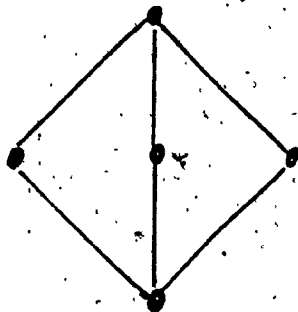
$$A = \{a \in B \mid b_1 \wedge a \leq b_2\}$$

$A$  is finite, since  $B$  is finite, so  $\vee A$  exists, and it is the greatest element of  $B$  satisfying the inequality

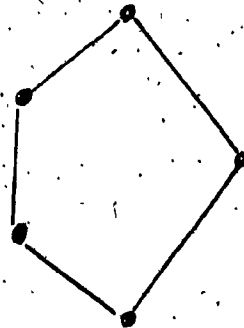
$$b_1 \wedge x \leq b_2$$

Hence  $\vee A$  is the pseudo complement of  $b_1$  relative to  $b_2$ , and since the least upper bound and the greatest lower bound of any two elements of  $B$  exist, we have proved that all the conditions for a pair to be a Pseudo-Boolean algebra is satisfied by  $(B, \leq)$ .

Lemma 2.6.3. A lattice is distributive if and only if it does not have a sublattice which can be represented by one of the following diagrams



[2.1]



[2.2]

Proof. See [B.1].

We conclude from the above three lemmas that a finite lattice is a Pseudo-Boolean algebra if and only if it does not contain a sublattice which can be represented by diagram [2.1] or diagram [2.2].

Now, we want to construct the Gentzen system of the intuitionistic propositional calculus in such a way that the completeness theorem holds, i.e., any formula in the intuitionistic propositional calculus is provable if and only if it is valid. To do so, we must set up the inference figures in such a way that the following condition is satisfied: "The lower sequent of any operational inference figure is not falsifiable if and only if the upper sequent, or the two upper sequents of the same inference figure are not falsifiable". We notice that the above condition is not satisfied by the inference figure schema  $\rightarrow \sim$  of section 2.2. As an example, consider the inference figure

$$\frac{P \rightarrow P}{\rightarrow \sim P, P}$$

where  $P$  is a propositional variable. Let  $(B, \leq)$  be the Pseudo-Boolean algebra represented by the diagram



and let  $h$  be any homomorphism such that  $h(P) = a$ . It is clear that the algebraic model  $(B, \leq, h)$  falsifies



the lower sequent of the inference figure, whereas the upper sequent is not falsifiable.

To avoid this difficulty, we get the Gentzen system of the intuitionistic propositional calculus from that of the classical propositional calculus by putting the following restriction: "The succedent of any sequent occurring in a proof does not contain more than one formula", and replacing the inference figure schema

$$\frac{\Gamma_1 \rightarrow \Theta_1, X, Y}{\Gamma_1 \rightarrow \Theta_1, X \vee Y}$$

by the following two inference figure schemata

$$\frac{\Gamma_1 \rightarrow \Theta_1, X}{\Gamma_1 \rightarrow \Theta_1, X \vee Y}$$

$$\frac{\Gamma_1 \rightarrow \Theta_1, Y}{\Gamma_1 \rightarrow \Theta_1, X \vee Y}$$

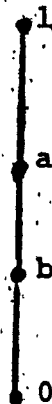
if the lower sequent of any two inference figures, which we get from the above two inference figure schemata, is not falsifiable, then the upper sequent of at least one of them is not falsifiable, which is sufficient for the completeness theorem to hold.

We cannot construct a method of search for counterexamples analogous to the one developed in section 2.3, since in the case of the intuitionistic propositional

calculus lemma 2.3.1 does not hold. As an example, consider the inference figure

$$\frac{P_1 \rightarrow P_2}{\rightarrow P_1 \supset P_2}$$

Let  $(B, \leq)$  be the Pseudo-Boolean algebra represented by the diagram



and let  $h$  be any homomorphism such that  $h(P_1) = a$ , and  $h(P_2) = b$ . It is clear that the algebraic model  $(B, \leq, h)$  falsifies the lower sequent of the inference figure, since

$$h(P_1 \supset P_2) = h(P_1) \Rightarrow h(P_2) = a \Rightarrow b = b \neq 1$$

whereas the upper sequent is satisfiable in  $(B, \leq, h)$ .

We have to develop a different method of search for counterexamples to non-valid formulae in the intuitionistic propositional calculus. That will be our task now.

Lemma 2.6.4. A formula  $X$  is falsifiable if and only if there is an algebraic model  $(B, \leq, h)$  such that one of the following conditions holds:

- (1)  $X$  is of the form  $Y \supset Z$  and  $h(Y) \not\leq h(Z)$ .
- (2)  $X$  is of the form  $Y \vee Z$  and  $h(Y) \neq 1 \neq h(Z)$ .
- (3)  $X$  is of the form  $Y \wedge Z$  and  $h(Y) \neq 1$  or  $h(Z) \neq 1$ .
- (4)  $X$  is of the form  $\sim Y$  and  $h(Y) \neq 0$ .

Proof. Any formula  $X$  has one of the forms,  $Y \supset Z$ ,  $Y \vee Z$ ,  $Y \wedge Z$  or  $\sim Y$  for some formulae  $Y, Z$ . We discuss the four cases.

Case 1.  $X$  has the form  $Y \supset Z$ . If  $X$  is falsifiable, then there exists an algebraic model  $(B, \leq, h)$  such that  $h(X) = h(Y) \supset h(Z) \neq 1$ , hence  $h(Y) \not\leq h(Z)$ ; for otherwise  $h(Y) \supset h(Z)$  would be equal to 1. Now, assume that there exists an algebraic model  $(B, \leq, h)$  such that  $h(Y) \leq h(Z)$ , hence  $h(Y) \supset h(Z) = 1$ , since  $h(Y) \wedge 1 = h(Y) \leq h(Z)$ , and so  $X$  is falsifiable.

Case 2.  $X$  has the form  $Y \vee Z$ . If  $X$  is falsifiable, then for some algebraic model  $(B, \leq, h)$  we have  $h(X) = h(Y) \vee h(Z) \neq 1$ , hence  $h(Y) \neq 1 \neq h(Z)$ . To prove the converse, let  $(B, \leq, h)$  be an algebraic model such that  $h(Y) \neq 1 \neq h(Z)$ , if  $h(Y) \vee h(Z) \neq 1$ , there is nothing to prove. If not, consider the algebraic model

$(B', \leq', h')$ , where  $B' = B \cup \{1'\}$ ,  $\leq' = \leq \cup \{(b, 1') : b \in B\}$  i.e.,  $1'$  is the maximal element of  $(B', \leq')$ , and  $h'$  is the homomorphism from  $F$  to  $(B', \leq')$ , which agrees with  $h$  on the set of all propositional variables. It is clear that  $h(Y) = h'(Y)$ , and  $h(Z) = h'(Z)$  hence  $h'(X) = h'(Y) \vee h'(Z) = h(Y) \vee h(Z) = 1 \neq 1'$  which means that  $X$  is falsifiable in  $(B', \leq', h')$ .

Case 3.  $X$  has the form  $Y \wedge Z$ . In this case it is clear that  $X$  is falsifiable if and only if there is an algebraic model  $(B, \leq, h)$ , such that  $h(Y) \neq 1$  or  $h(Z) \neq 1$ .

Case 4.  $X$  has the form  $\sim Y$ . If  $X$  is falsifiable, then there is an algebraic model  $(B, \leq, h)$  such that  $h(X) = h(\sim Y) = -h(Y) \neq 1$ , implies  $h(Y) \neq 0$ , since  $-0 = 1$ .

Conversely, if there exists an algebraic model  $(B, \leq, h)$  such that  $h(Y) \neq 0$ , then  $h(X) = -h(Y) \neq 1$ , (since  $h(Y) \wedge 1 = h(Y) \neq 0$ , whereas  $h(Y) \wedge -h(Y) = 0$ ), and so  $X$  is falsifiable in  $(B, \leq, h)$ .

This completes the proof of the lemma.

Now, let  $X$  be any formula. We can, using lemma 2.6.4 and similar techniques, write down all the conditions which must be satisfied by any algebraic model  $(B, \leq, h)$  to be a counterexample to  $X$ . If these conditions are not self-contradictory and do not contradict the axioms of Pseudo-Boolean algebras, we shall be able to draw one or

more diagrams which satisfy them and can be completed to a Pseudo-Boolean algebra  $(B_0, \leq_0)$  say, and so  $(B_0, \leq_0, h_0)$  will be a counterexample to  $X$ , where  $h_0$  is any homomorphism from  $F$  to  $(B_0, \leq_0)$ , such that it assigns to each propositional variable  $P$  occurring in  $X$  the value  $h(P)$ .

If the conditions are self-contradictory or contradict the axioms of Pseudo-Boolean algebras,  $X$  will not be falsifiable in any algebraic model; in other words,  $X$  is valid.

We cannot be more precise since the search for counterexamples to any formula  $X$  depends on its construction, but the examples we give below clarify the method.

Example 1. Let  $X$  be the formula

$$P \supset \sim P.$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is

$$h(P) \not\leq \sim h(P).$$

But this condition cannot be satisfied by any algebraic model, since we always have

$$h(P) \wedge \sim h(P) = 0,$$

and so  $h(P) \leq -h(P) \Rightarrow 0 = -h(P)$ , hence  $X$  is valid.

Example 2: Let  $X$  be the formula

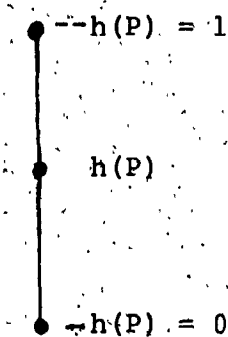
$$\neg P \supset P$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is

$$\neg h(P) \not\leq h(P) \quad (1)$$

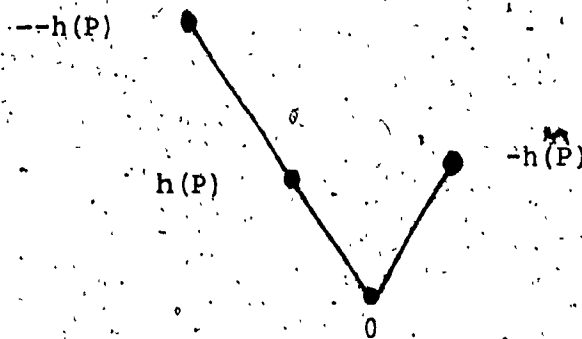
We have two cases:

Case 1.  $-h(P) \leq h(P)$ . In this case  $-h(P)$  must be equal to zero, which implies that  $h(P)$  is equal to one, and we get the following counterexample:



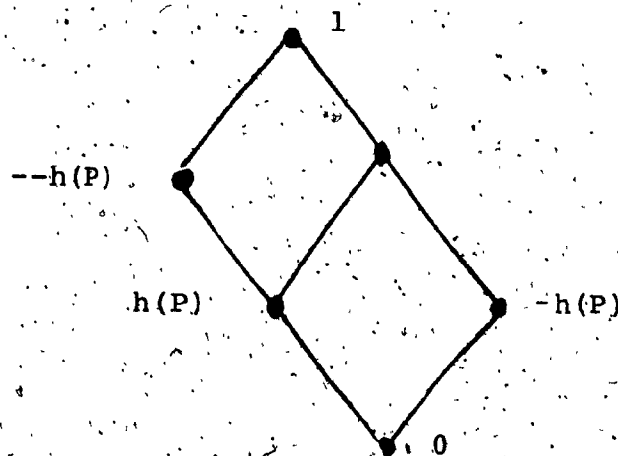
[3.1]

Case 2.  $-h(P) \not\leq h(P)$ , and in this case, condition (1) will be satisfied by



[3.2]

which can be completed to the counterexample given by diagram [3.3].



[3.3]

Example 3. Let  $X$  be the formula

$$(P_1 \supset P_2) \supset (\sim P_2 \supset \sim P_1).$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is

$$(h(P_1) \Rightarrow h(P_2)) \not\leq (-h(P_2) \Rightarrow -h(P_1)) \quad (1)$$

But in any algebraic model  $(B, \leq, h)$ , we have

$$\begin{aligned} h(P_1) \wedge ((h(P_1) \Rightarrow h(P_2)) \wedge -h(P_2)) \\ &= (h(P_1) \wedge (h(P_1) \Rightarrow h(P_2))) \wedge -h(P_2) \\ &\leq h(P_2) \wedge -h(P_2) \\ &= 0 \end{aligned}$$

Thus we have

$$-h(P_2) \wedge (h(P_1) \Rightarrow h(P_2)) \leq -h(P_1)$$

Hence

$$h(P_1) \Rightarrow h(P_2) \leq -h(P_2) \Rightarrow -h(P_1)$$

and so condition (1) is not satisfied by any algebraic model, and  $X$  is valid.

Example 4. Let  $X$  be the formula

$$(\sim P_2 \supset \sim P_1) \supset (P_1 \supset P_2)$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is

$$h(\sim P_2 \supset \sim P_1) \not\leq h(P_1 \supset P_2)$$



But this condition is satisfied only if

$$h(P_1) \leq h(P_2)$$

for otherwise we have

$$h(P_1 \supset P_2) = 1.$$

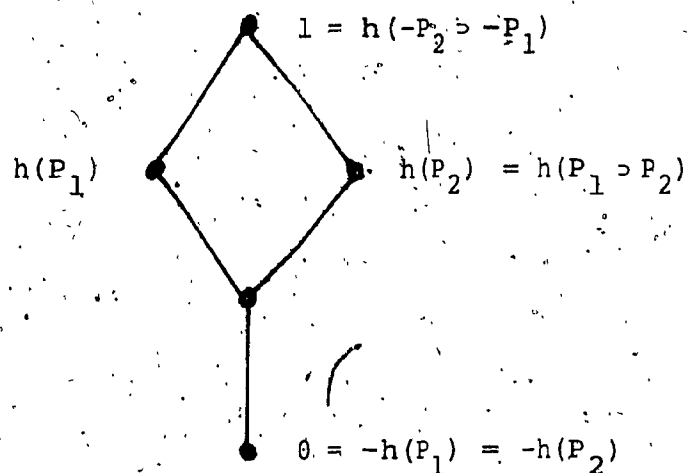
Now, we have two cases:

Case 1.  $h(P_2) < h(P_1)$ . In this case we get the following counterexample:

$$\begin{array}{l} 1 = h(\neg P_2 \supset \neg P_1) \\ h(P_1) \\ h(P_2) = h(P_1 \supset P_2) \\ 0 = -h(P_1) = -h(P_2) \end{array}$$

[4.1]

Case 2.  $h(P_1)$  and  $h(P_2)$  are not comparable and in this case we get the following counterexample:



[4.2]

Any other counterexample to  $X$  will be a superstructure of one of the above two since if  $h(P_1) \wedge h(P_2) = 0$ , we do not get any counterexample to  $X$ .

Example 5. Let  $X$  be the formula

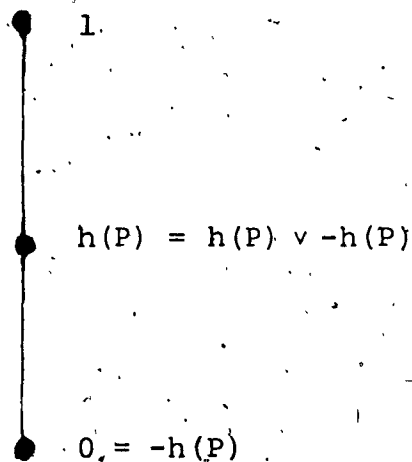
$$p \vee \sim p$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is

$$h(P) \neq 1 \neq h(\sim P) = -h(P)$$

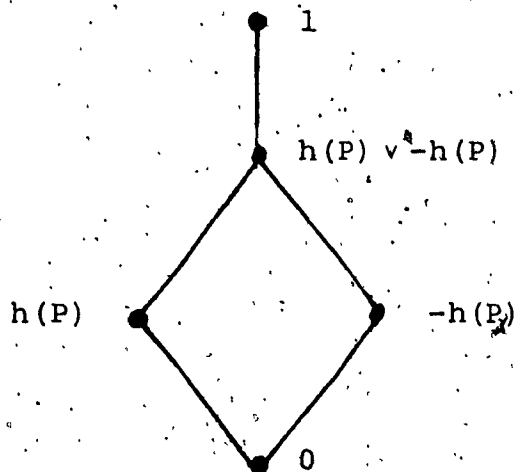
We have two cases.

Case 1.  $-h(P) < h(P)$  . In this case  $-h(P)$  must be equal to zero, and we get the following counterexample:



[5.1]

Case 2.  $-h(P) \not< h(P)$  . In this case, we get the following counterexample.



[5.2]

Any other counterexample to X will be a superstructure of one of the above two.

Example 6. Let  $X$  be the formula

$$\sim((\sim P_2 \supset \sim P_1) \supset (P_1 \supset P_2))$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is,

$$h((\sim P_2 \supset \sim P_1) \supset (P_1 \supset P_2)) \neq 0$$

Now, let  $(B, \leq)$  be any Pseudo-Boolean algebra, and let  $h$  be any homomorphism from the set of formulae to  $(B, \leq)$  such that  $h(P_2) \neq 0$ .

We have

$$h(P_1 \supset P_2) = h(P_1) \supset h(P_2) \neq 0,$$

since

$$h(P_1) \wedge h(P_2) \leq h(P_2)$$

and

$$h(P_2) \neq 0$$

and so we have

$$h((\sim P_2 \supset \sim P_1) \supset (P_1 \supset P_2)) = h(\sim P_2 \supset \sim P_1) \supset h(P_1 \supset P_2)$$

$$= (-h(P_2) \supset -h(P_1)) \supset (h(P_1) \supset h(P_2))$$

$$\neq 0$$

since

$$(-h(P_2) \supset -h(P_1)) \wedge (h(P_1) \supset h(P_2)) \leq (h(P_1) \supset h(P_2))$$

and

$$h(P_1) \Rightarrow h(P_2) \neq 0.$$

Hence  $(B, \leq, h)$  is a counterexample to  $X$ .

Example 7. Let  $X$  be the formula

$$(\sim P_1 \vee \sim P_2) \supset \sim (P_1 \wedge P_2).$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is

$$h(\sim P_1 \vee \sim P_2) \neq h(\sim (P_1 \wedge P_2)).$$

Hence, we must have

$$(\neg h(P_1) \vee \neg h(P_2)) \neq \neg (h(P_1) \wedge h(P_2)). \quad (1)$$

But, in any algebraic model  $(B, \leq, h)$ , we have

$$\begin{aligned} (h(P_1) \wedge h(P_2)) \wedge (\neg h(P_1)) &= (h(P_2) \wedge h(P_1)) \wedge (\neg h(P_1)) \\ &= h(P_2) \wedge (h(P_1) \wedge (\neg h(P_1))) \\ &= h(P_2) \wedge 0 \\ &= 0 \end{aligned}$$

and so we have

$$\neg h(P_1) \leq \neg (h(P_1) \wedge h(P_2)) \quad (2)$$

By the same argument, we get

$$\neg h(P_2) \leq \neg(h(P_1) \wedge h(P_2)) \quad (3)$$

From (2) and (3), we get

$$\neg h(P_1) \vee \neg h(P_2) \leq \neg(h(P_1) \wedge h(P_2))$$

and so condition [1] is not satisfied in any algebraic model, thus  $X$  is valid.

Example 8. Let  $X$  be the formula

$$\neg(P_1 \wedge P_2) \supset (\neg P_1 \vee \neg P_2)$$

The condition which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  is

$$h(\neg(P_1 \wedge P_2)) \not\leq h(\neg P_1 \vee \neg P_2)$$

Hence, we must have

$$\neg(h(P_1) \wedge h(P_2)) \not\leq \neg h(P_1) \vee \neg h(P_2) \quad [1]$$

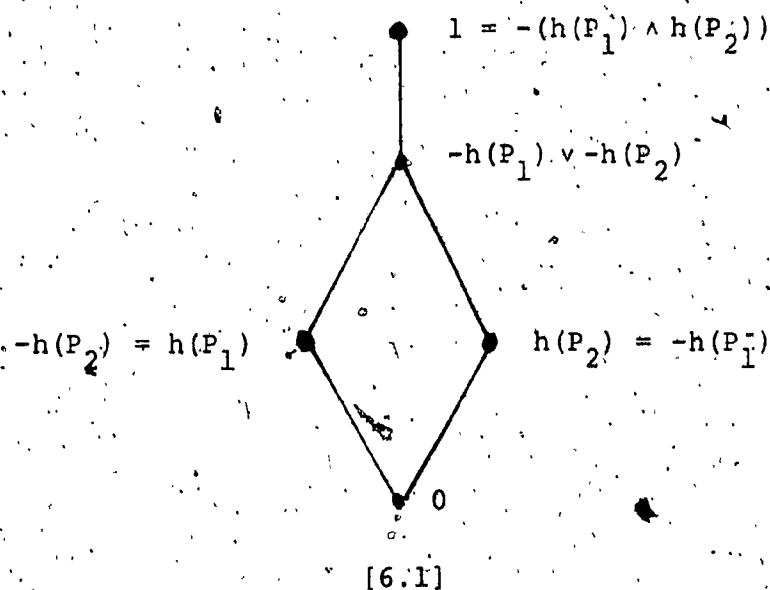
Condition [1] is not satisfied if  $h(P_1)$  and  $h(P_2)$  are comparable, since if  $h(P_1) \leq h(P_2)$ , say, we get

$$\neg(h(P_1) \wedge h(P_2)) = \neg h(P_1) \leq \neg h(P_1) \vee \neg h(P_2)$$

Now, assume that  $h(P_1)$  and  $h(P_2)$  are not comparable.

We have two cases.

Case 1.  $h(P_1) \wedge h(P_2)$  is equal to zero. In this case, we get the following counterexample



Case 2.  $h(P_1) \wedge h(P_2)$  is not equal to zero. We shall deal with this case in the next example.

Example 9. Let  $X$  be the formula

$$(\sim(P_1 \wedge P_2) \supset (\sim P_1 \vee \sim P_2)) \vee P_1 \supset (\sim P_1 \vee \sim P_2)$$

The conditions which must be satisfied by  $(B, \leq, h)$  to be a counterexample to  $X$  are

$$(1) \quad h(\sim(P_1 \wedge P_2) \supset (\sim P_1 \vee \sim P_2)) \neq 1$$

$$(2) \quad h(P_1 \supset (\sim P_1 \vee \sim P_2)) \neq 1$$

which are satisfied if and only if the conditions

$$(3) \quad \neg(h(P_1) \wedge h(P_2)) \neq \neg h(P_1) \vee \neg h(P_2)$$

$$(4) \quad h(P_1) \neq -h(P_1) \vee -h(P_2)$$

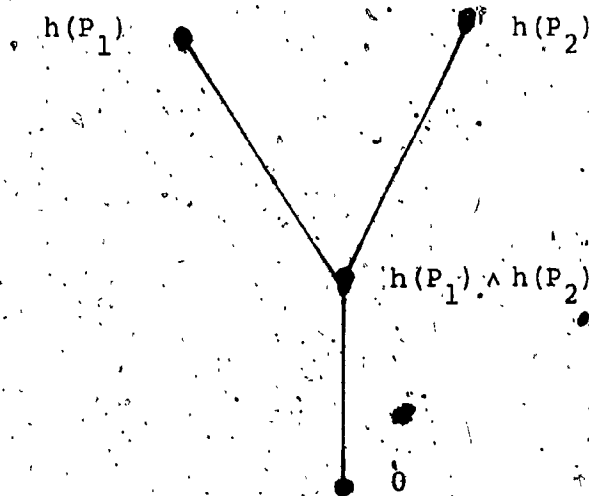
are satisfied. But condition (4) is satisfied only if

$$h(P_2) \wedge h(P_1) \neq 0$$

for otherwise we have,

$$h(P_1) \leq -h(P_2)$$

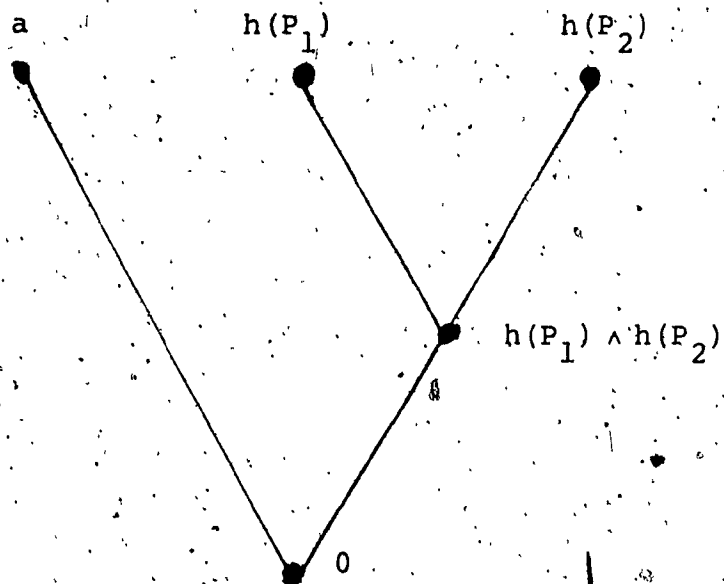
So we consider the diagram



[7.1]

If  $-(h(P_1) \wedge h(P_2))$  is equal to zero, then condition (3) is not satisfied, so we get the diagram





[7.2]

If for every  $a$  such that

$$(h(P_1) \wedge h(P_2)) \wedge a = 0$$

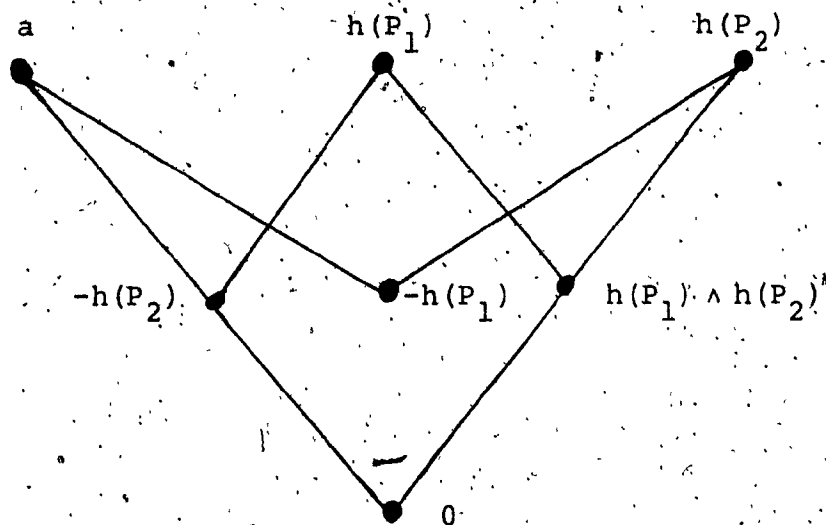
we have

$$a \wedge h(P_1) = 0 \quad \text{and} \quad a \wedge h(P_2) = 0$$

we shall get

$$-h(P_1) = -h(P_2) = -(h(P_1) \wedge h(P_2))$$

and condition (3) will not be satisfied. So we consider the following diagram



[7.3]

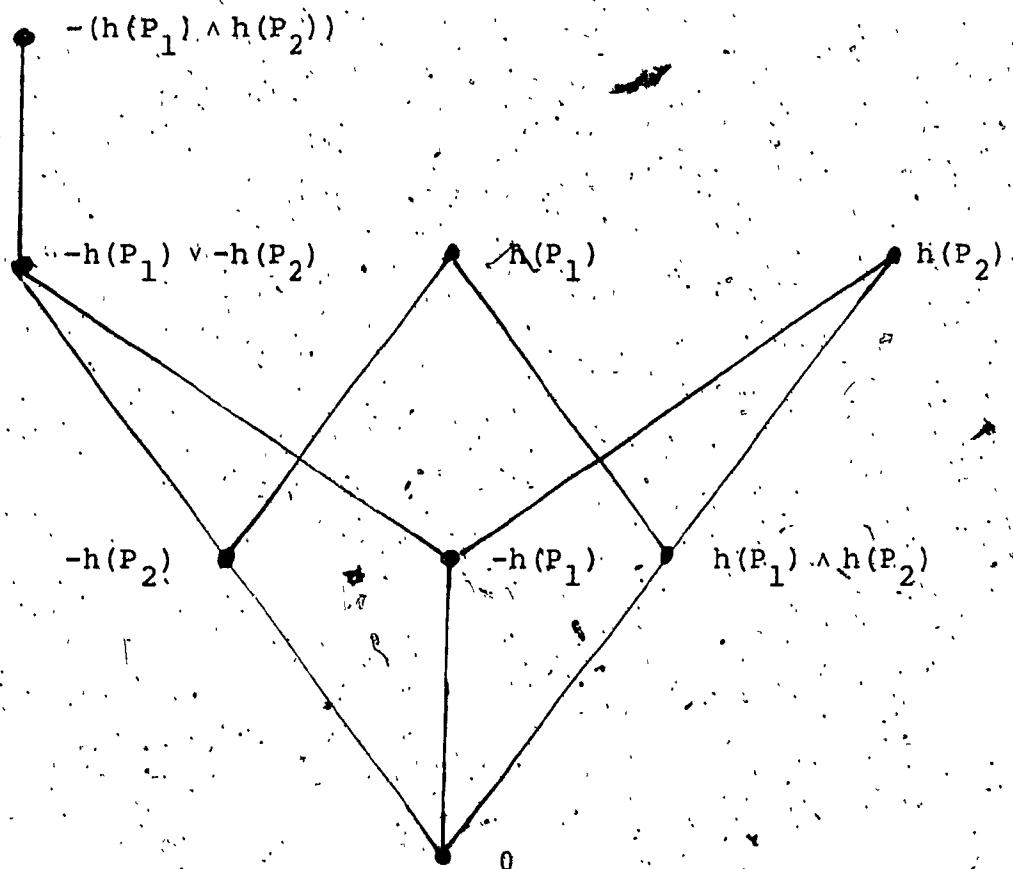
Now, we have

$$-h(P_2) \vee -h(P_1) = a$$

But we must have

$$\neg(h(P_1) \wedge h(P_2)) \neq -h(P_1) \vee -h(P_2)$$

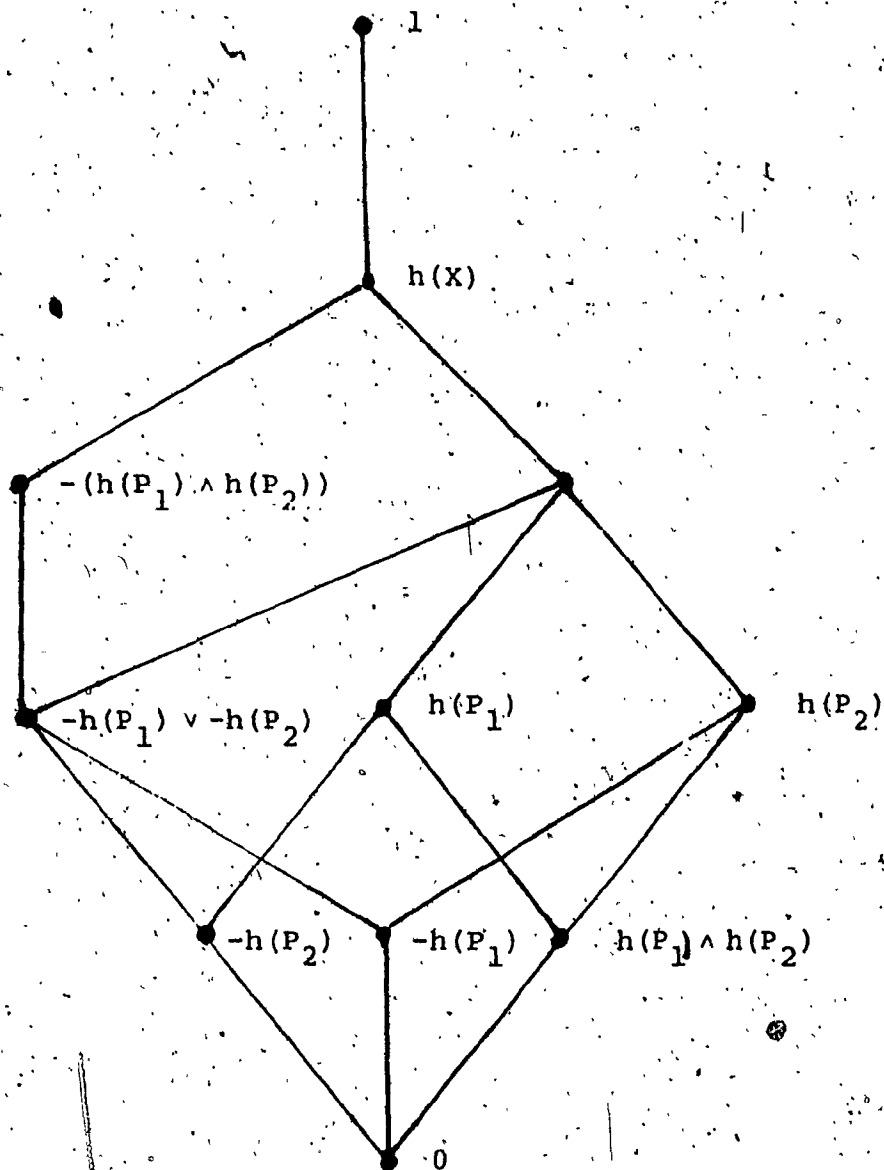
So we consider the following diagram



[7.4]

It is clear the diagram [7.4] satisfies conditions (3) and (4), thus it satisfies conditions (1) and (2).

Diagram [7.4] can be completed to the counterexample given by diagram [7.5]



[7.5]

Any other counterexample to  $X$  will be a superstructure of the one given by diagram [7.5].

The counterexample which we get in case 2 of Example 8. is the one given by diagram [7.5].

## CHAPTER III

### THE DECIDABILITY OF THE CLASSICAL AND THE INTUITIONISTIC PROPOSITIONAL CALCULI

#### 3.1. Introduction

By a decision procedure for a given formal logical system  $LS$ , we understand a mechanical method which permits us to decide, in each particular case, in a finite period of time, whether or not a given formula, which belongs to the language of  $LS$ , can be recognized as valid in the semantics of  $LS$ . By a mechanical method, we mean a method which can be carried out by a computing machine. The decision problem for  $LS$  is the problem of determining whether or not a decision procedure for  $LS$  exists. A formal logical system  $LS$  is called decidable (undecidable), if the solution of the decision problem for  $LS$  is positive (negative). The recognition of the decision problems for formal logical systems goes back to Schroder 1895, Lowenheim 1915, and Hilbert 1918.

In this chapter, we shall prove the decidability of the classical and the intuitionistic propositional calculi. In each case, we shall give two proofs: one is proof-theoretic, and the other is model-theoretic.

The two proof-theoretic proofs for the decidability of the classical and the intuitionistic propositional calculi, may be found in ([Sz.]pp.103-106).

### 3.2. A Proof-Theoretic Proof for the Decidability of the Classical and the Intuitionistic Propositional Calculi.

We start this section by the following two theorems.

Theorem 3.2.1. Every proof in the classical propositional calculus can be transformed into another proof in the classical propositional calculus with the same endsequent in which no cuts occur.

Proof. See ([Sz.], paper 3, Section III, § 3).

Theorem 3.2.2. Every proof in the intuitionistic propositional calculus can be transformed into another proof in the intuitionistic propositional calculus with the same endsequent, in which no cuts occur.

Proof. See ([Sz.], paper 3, Section III, § 3).

We conclude from Theorem 3.2.1 (Theorem 3.2.2) that we can find a proof for every provable formula in the classical propositional calculus (the intuitionistic propositional calculus) in which no cuts occur.

Corollary 3.2.1. There exists a proof for every provable formula  $X$  in the classical propositional calculus such that any formula  $Y$  which occurs in the proof is a subformula of  $X$ .

Proof. We notice that if  $Y$  is any formula which occurs in the upper sequent of a structural inference figure, other than a cut, then  $Y$  occurs in the lower sequent of the inference figure. Also, if  $Y$  occurs in the upper sequent, or in one of the two upper sequents, of an operational inference figure, then  $Y$  will occur in the lower sequent of the inference figure, or  $Y$  will be a subformula of a formula  $Z$  which occurs in the lower sequent of the inference figure.

Now, let  $X$  be any provable formula in the classical propositional calculus. We know that we can find a proof for  $X$  in which no cuts occur. Assume that a formula  $Y$  occurs in that proof, i.e., the proof contains a sequent  $S$ , say, in which  $Y$  occurs. We get the endsequent from  $S$  by consecutive applications of structural inference figures, other than cuts, and operational inference figures. So  $Y$  occurs in the endsequent or  $Y$  is a subformula of a formula occurring in the endsequent. But the endsequent contains only one formula, namely  $X$ . Hence  $Y$  is a subformula of  $X$ . This completes the proof of the corollary.

We can apply the above argument to the intuitionistic propositional calculus to get the following corollary of Theorem 3.2.2.

Corollary 3.2.2. There exists a proof for every provable formula  $X$  in the intuitionistic propositional calculus such that any formula  $Y$  which occurs in the proof is a subformula of  $X$ .

A reduced sequent is a sequent in whose antecedent no formula occurs more than two times, and in whose succedent, furthermore, one and the same formula occurs no more than two times.

Now, we prove the following two lemmas which we shall use in the proof of the decidability of the classical and the intuitionistic calculi.

Lemma 3.2.1. There exists a proof for every provable formula  $X_1$  in the classical propositional calculus consisting only of reduced sequents, such that any formula  $Y_1$  which occurs in the proof is a subformula of  $X_1$ .

Proof. Let  $T$  be a proof for  $X_1$  such that any formula  $Y_1$  which occurs in  $T$  is a subformula of  $X_1$ , and let  $S$  be any sequent occurring in  $T$ . We get  $S'$ , which is called a "reduction instance of  $S$ " as follows:

- (1) If  $S$  is an axiom or the endsequent, then  $S'$  is the same as  $S$ .



(2) If  $S$  is the upper sequent of an inference figure, then we eliminate each formula which occurs in the antecedent from  $\Gamma_1$  (or  $\Gamma_1$  and  $\Gamma_2$ ) as many times as is necessary to ensure that it occurs in the antecedent no more than twice, and if it occurs exactly twice, then at least one of the occurrences must be as the  $X$  or the  $Y$  of the inference figure (i.e., the formulae that are designated by  $X$  or  $Y$  in the inference figure schemata). We do the same for the succedent with  $\Theta_1$  and  $\Theta_2$  instead of  $\Gamma_1$  and  $\Gamma_2$ .

Now assume that the lower sequent of the inference figure (whose upper sequent is  $S$ ) is  $S_1$ . It is clear that when we apply the same inference figure schema on  $S'$ , we get  $S'_1$  or a sequent from which  $S'_1$  can be derived by means of thinnings, contractions, and interchanges, such that in the course of this operation only reduced sequents occur.

This means that  $T$  can be transformed to a proof  $T'$ , to  $X_1$ , where  $T'$  satisfies the conditions of the lemma.

Lemma 3.2.2. There exists a proof for every provable formula  $X_1$  in the intuitionistic propositional calculus consisting only of reduced sequents, such that any formula  $Y_1$  which occurs in the proof is a subformula of  $X_1$ .

This lemma can be proved in the same way as lemma 3.2.1.

Now we are ready to prove the decidability of the classical and the intuitionistic propositional calculi.

Theorem 3.2.3. The classical propositional calculus is decidable.

Proof. The decision procedure can be described as follows: For any formula of the classical propositional calculus, consider the set  $D$  of all reduced sequents in which only subformula of  $X$  may occur. (There are only a finite number of these sequents, since the number of the subformulae of  $X$  is finite, and no formula can occur in the same sequent more than four times.) We investigate which of these are axioms, then we examine each of the remaining sequents to determine whether there exists an inference figure in which there occur as upper sequents one or two of the axioms, and in which the sequent in question is the lower sequent (we have to check finitely many inference figures since  $D$  contains finitely many axioms). If this is the case, the sequent is called provable. We repeat the above process for the remaining sequents, but in this case we search for an inference figure which has one or two of the axioms or the sequents that have already been found to be provable as upper sequents and the sequent in question as the lower sequent. (In this case too, we have to check

finitely many inference figures.) We continue in this process until the sequent  $\rightarrow X$ , turns out to be provable, i.e.,  $X$  is provable, or until the procedure yields no new provable sequents.

So, we have a mechanical method which permits us to decide, for every formula  $X$ , in a finite period of time, whether or not  $X$  is valid, (since by the completeness theorem,  $X$  is provable if and only if  $X$  is valid), and so the classical propositional calculus is decidable.

Theorem 3.2.4. The intuitionistic propositional calculus is decidable.

Proof. The proof is the same as the one of Theorem 3.2.3, but in this case  $D$  will be the set of all reduced sequents in which only subformula of  $X$  may occur, and whose succedents contain at most one formula.

We notice that the method which has been given in section 2.5 is not a decision procedure for the classical predicate calculus despite its giving us counterexample to every non-valid formula (i.e., permits us to decide whether or not any formula in the classical predicate calculus is valid) since the process of reading a counterexample off an infinite path is not mechanical, and takes human intelligence to be executed.

### 3.3. A Model-Theoretic Proof for the Decidability of the Classical Propositional Calculus.

We start with the following lemma.

Lemma 3.3.1. If  $f_1$  and  $f_2$  are any two homomorphisms from the set of all formulae  $F$  to  $2$ , such that they agree on each propositional variable which occurs in a formula  $X$ , then  $f_1(X) = f_2(X)$ .

The proof is by induction on the number of logical connectives in  $X$ .

But Theorem 1.5.1 states that any formula in the classical propositional calculus is valid if and only if  $f(X) = 1$ , for each homomorphism  $f$  from  $F$  to  $2$ . Combining lemma 3.3.1 and Theorem 1.5.1 together, we get the following decision procedure:

Let  $X$  be any formula in the classical propositional calculus, and let  $P_X$  be the set of all propositional variables occurring in  $X$ . As both  $P_X$  and  $2$  are finite, we have finitely many functions  $f_1, \dots, f_n$ , say, from  $P_X$  to  $2$ . Now, for each  $i, 1 \leq i \leq n$ , let  $f_i^*$  be the homomorphism which agrees with  $f_i$  on  $P_X$  and gives the value zero to each propositional variable which is not an element of  $P_X$ . We calculate the value of  $X$  under  $f_i^*$ ,  $1 \leq i \leq n$ . If it is equal to 1 for each  $f_i^*$ , then  $X$  is valid, but if for some  $i$ , we get  $f_i^*(X) = 0$ , then  $X$  is

not valid. (This is sufficient to determine the validity of  $X$  since any other homomorphism from  $F$  to  $2$  will agree with one of the  $f_i$ 's on  $P_X$ .) It is easy to see that the above process can be carried out in a finite period of time by a computing machine.

### 3.4. A Model-Theoretic Proof for the Decidability of the Intuitionistic Propositional Calculus.

We start with the following two lemmas.

Lemma 3.4.1. If  $h_1$  and  $h_2$  are any two homomorphisms from the set of all formulae  $F$  to any Pseudo-Boolean algebra such that they agree on each propositional variable occurring in a formula  $X$ , then  $h_1(X) = h_2(X)$ .

The proof is by induction on the number of the logical connectives in  $X$ .

Lemma 3.4.2. If  $B_1$  and  $B_2$  are two isomorphic Pseudo-Boolean algebras, then for any formula  $X$ ,  $B_1 \models X$  if and only if  $B_2 \models X$ .

Proof. Clear.

Now, combining lemmas 3.4.1 and 3.4.2 together with Theorem 1.10.1 (which states that a formula  $X$  is valid if and only if it is valid in each Pseudo-Boolean algebra with at most  $2^{2^Y}$  elements), we get the following decision procedure.

Let  $X$  be any formula, and  $\gamma$  be the number of the subformulae of  $X$ , for each  $n \leq 2^{2^\gamma}$  consider the set  $\{0, \dots, n\}$  and let  $B_{n_1}, \dots, B_{n_m}$  be all the different Pseudo-Boolean algebras defined on it. (The number of all Pseudo-Boolean algebras defined on  $\{0, \dots, n\}$  is finite since we can define an ordering relation, and join and meet operations on  $\{0, \dots, n\}$  only in finitely many different ways.) Now, we apply a decision procedure similar to the one in the previous section, with  $2$  replaced by each one in turn of the  $B_{n_i}$ 's to determine whether or not  $X$  is valid in each  $B_{n_i}$ ,  $1 \leq i \leq m$ , and  $X$  is valid if and only if  $X$  is valid in each  $B_{j_i}$ ,  $0 \leq j \leq 2^{2^\gamma}$ ,  $1 \leq i \leq m$ . (This is sufficient to determine the validity of  $X$  since any other Pseudo-Boolean algebra with at most  $2^{2^\gamma}$  elements will be isomorphic to one of the  $B_{j_i}$ 's.)

So, we have a mechanical method which permits us to decide, for every formula  $X$ , in a finite period of time whether or not  $X$  is valid. In other words, we have a decision procedure for the intuitionistic propositional calculus.

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